

Alpha-congruence for dispersive billiards

KARI ELORANTA†

Institute of Mathematics, Helsinki University of Technology, 02150 Espoo, Finland

(Received 1 June 1989 and revised 30 April 1990)

Abstract. We show the stability in the sense of α -congruence of dispersive (Sinai) planar billiards that are Bernoulli flows. The perturbations are either billiards on slightly altered tables or geodesic flows on nearby manifolds.

0. Introduction

In this paper we establish perturbation results for certain billiard systems. In the first part we will show that for a large class of dispersive plane billiards the measure theoretic entropy of the flow in the phase space is continuous under smooth perturbation of the curved boundaries of the billiard table. We then extend this to a new type of stability which is called α -congruence and discuss its qualitative features as well as generalizations of the result. This concept was introduced by Ornstein and Weiss in [OW]. Intuitively the notion is as follows: two dynamical systems are α -congruent if they are measure theoretically isomorphic and the isomorphism moves all but α of the state space by less than α . The formulation is useful since it enables one to unify the stability analysis of diverse dynamical systems of both deterministic and stochastic origin (as indicated in [OW] and [El]).

In the second part (§§ 5-7) we investigate a class of perturbations of a dispersive billiard originally proposed by V. I. Arnold. Here the perturbation is a geodesic flow on a surface of non-positive curvature and we again show the convergence of entropies as well as α -congruence of the original system and the perturbation. Comparisons to structural stability and variations of it are made.

1. The billiard flow

Let Q be a bounded and connected plane domain which we call the billiard table. We shall assume that the boundary of Q consists of a finite number of C^3 curves $\{\partial Q_i\}$ which intersect each other transversally and only at their endpoints. Consequently for every $q \in \partial Q$ we have a unit interior normal vector $n(q)$ except if q is a point of intersection.

Definition 1.1. By a *billiard* in Q we mean a dynamical system generated by a uniform linear motion of a point-mass within the domain Q , assumed to be reflected by the boundary of the domain according to the law 'angle of incidence equals angle of reflection'. By *dispersive billiard* we mean a billiard on a domain Q with non-positive geodesic curvature at every point on ∂Q where $n(q)$ exists and negative

† Research partially supported by the Academy of Finland and the Emil Aaltonen Foundation.

curvature somewhere on ∂Q . Hence the left and right limits of the curvature exist everywhere on the boundary. We denote the curved boundary (*the scatterer*) by $\partial\hat{Q}$.

For any $q \in Q$ let $\Omega(q)$ be the set of directions at q that point into Q . The *phase space* M of the billiard is the skew product of Q with $\Omega(\cdot)$. This set consists of all the possible positions and directions of motion that the particle can have. It is a compact three-dimensional manifold with boundary and a subset of the unit tangent bundle SQ . By π we denote the natural projection from M onto Q . We denote the components of ∂M by $\partial M_i = \pi^{-1}(\partial Q_i)$ and define $\partial\hat{M} = \pi^{-1}(\partial\hat{Q})$. Metrize M by the Euclidean metric and denote this metric by D .

On ∂M_i we have natural coordinates (r, ϕ) where r is the arclength and ϕ is the angle measured counterclockwise from $n(q(r))$, $\phi \in [-\pi/2, \pi/2]$. Let

$$G = \{x \in \partial M \mid \langle x, n(\pi(x)) \rangle > 0\},$$

$$B_1 = \{x \in \partial M \mid \langle x, n(\pi(x)) \rangle = 0\} \quad \text{and}$$

$$B_2 = \{x \in \partial M \mid \pi(x) \in \partial Q_i \cap \partial Q_j \text{ some } i, j, i \neq j\}.$$

The set $B = B_1 \cup B_2$ is called *the set of singular points*.

Let μ be the Lebesgue measure on M and denote the one-parameter group of shifts along orbits of the billiard by $\{S_t \mid t \in \mathbb{R}\}$. S_t preserves μ hence the dynamical system $(M, \mathcal{B}(M), S_t, \mu)$ is a flow. Let $M_B = \{x \in M \mid S_t x \in B \text{ for some } t \in \mathbb{R}\}$. It can be shown that $\mu(M_B) = 0$.

Let $\bar{\tau}(x) = \inf\{t > 0 \mid S_t x \in G \cap \partial\hat{M}\}$. A billiard is said to have *finite horizon* if $\exists \bar{\tau} > 0$ such that $\forall x \in M \bar{\tau}(x) \leq \bar{\tau}$. Also define the minimum distance between two scatterers by letting $\tau = \inf_{i \neq j} d(\partial\hat{Q}_i, \partial\hat{Q}_j)$. Typical dispersive billiard tables that also have a finite horizon are illustrated in figure 1.

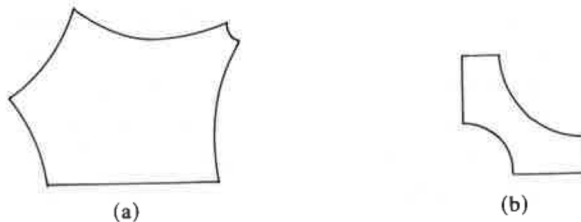


FIGURE 1

We now define a class of billiards that will be of main interest in the first half of this paper.

Definition 1.2. A *canonical billiard table* (c.b.t.) is a rectangle, possibly with dispersive but mutually disjoint outer boundary components and a finite number of disjoint convex obstacles in the interior. All boundaries are reflecting and the geometry is such that the billiard flow has finite horizon. Curvature of the dispersive pieces is strictly negative everywhere even in one-sided limits at the endpoints. Finally all inside angles between boundary components are right.

A simple example of a c.b.t. is given in figure 1(b). Note that this definition implies $0 < \tau < \bar{\tau} < \infty$. Certain generalizations beyond c.b.t. are discussed at the end of § 3.

1.3. From a billiard on a c.b.t. a *toral billiard* can be constructed through reflection argument as indicated in [S1]. There it was also shown that the mixing properties and entropy of the original and toral system are identical. The same trick enables us to extend all our results for billiards on canonical tables to corresponding toral billiards. Finally we note that although c.b.t. is semidispersive the toral extension is dispersive i.e. the curvature is negative at every boundary point.

2. Ergodic properties, Lyapunov exponents and the entropy formula

In this section we briefly describe the ergodic properties of dispersive billiards. In particular we elaborate on the fibre curvature and show the formula for the entropy of the flow. We only discuss details if they are pertinent to later sections and no proofs are included. They can be found in the main reference or in the ones given.

2.1. Let T be the *derived automorphism* of the billiard flow i.e. the restriction of S_t into ∂M . Let $x \in U \subset \partial M$, U an open neighborhood. By a *locally contracting transversal fibre* (l.c.t.f.) of x we mean a curve in U consisting of points y such that $d(T^i x, T^i y) \rightarrow 0$ as $i \rightarrow \infty$. Theorems 3.1 and 4.1 in [S1] together imply that for ν -almost all $x_0 \in G$ there exists a C^1 curve γ through x_0 belonging to the l.c.t.f. of x_0 , given by an equation $\phi = \phi(r)$:

$$\frac{d\phi}{dr} = \kappa^{(c)}(x(r, \phi)) \cos \phi.$$

Here $\kappa^{(c)}$ is the curvature of fibre. The length of γ between break points is bounded from below by a positive function $C(x)$. Moreover by taking $x_i = T^i x_0$, $i \in \mathbb{N}$, $x_0 \in G$ we construct l.c.t.f. $\gamma^{(i)}$ containing x_i as above. Then define by $\Gamma^{(c)}(x_0) = \bigcup_{i=0}^{\infty} T^i \gamma^{(i)}$ the *complete contracting transversal fibre* for T at x_0 . By using an analogous definition of l.c.t.f. for S_t a similar construction can be performed yielding the l.c.t.f. through any $x \in M$ and its global counterpart. The complete contracting transversal fibre for S_t is a piecewise differentiable curve in M its singularities being caused by corners of the table or tangential intersections of the trajectory with ∂Q . The contracting transversal fibre has a natural counterpart, the *expanding transversal fibre* $\Gamma^{(e)}(x)$. They are preserved by the flow i.e. $\Gamma^{(\cdot)}(S_t x) = S_t \Gamma^{(\cdot)}(x)$. The fibre structures $\{\Gamma^{(\cdot)}(x)\}_{x \in M}$ are called the *transversal fields* of S_t (T similarly).

For μ -almost every $x \in M$ the contraction rate along $\Gamma^{(c)}(x)$ is given by $\kappa^{(c)}(x)$. Its explicit form is

$$-\kappa^{(c)}(x) = \frac{1}{\tau_1 + \frac{1}{-\frac{2k_1(x)}{\cos \phi_1(x)} + \frac{1}{\tau_2 + \frac{1}{-\frac{2k_2(x)}{\cos \phi_2(x)} + \dots}}} \quad (2.1)$$

where the τ_i 's are time intervals between two consecutive reflections of the trajectory of x from the scatterer. $k_i(x) < 0$ is the curvature of the boundary and $\phi_i \in [-\pi/2, \pi/2]$ the angle of incidence of the i th point of reflection from the scatterer.

Let $\kappa_i(x)$ denote the i th convergent of $-\kappa^{(c)}(c)$. From this point on we use the convention $\kappa(x) = -\kappa^{(c)}(x)$. The formula is presented in [S1] using slightly different notation. It follows from considering the two basic mechanisms in the billiard flow. If k_0 and k_t are the curvatures of a section of a raybundle in the beginning and at time t and no reflections have taken place in between we have the *focusing relation*:

$$\frac{1}{k_0} = \frac{1}{k_t} + t. \quad (2.2)$$

If k_- and k_+ denote the curvatures immediately before and after a reflection then we have the *dispersion relation*:

$$k_+ = k_- + \frac{2k}{\cos \phi}. \quad (2.3)$$

Applying these to the evolution of a section $\gamma \subset M$ of an infinitesimal ray bundle on Q (the fibre through x) yields the formula (2.1). Let $\pi(\gamma) \subset \partial Q_i$, some i and let γ be given by $\phi = \phi(r)$. If after m reflections from the scatterer $S_t \gamma$ has zero curvature then the required initial curvature of γ is given by

$$\frac{d\phi}{dr} = \kappa_{2m}(x) \cos \phi(r).$$

γ is assumed to reflect from one boundary piece at a time and not to have tangential rays. To obtain the curvature of the asymptotically parallel ray bundle we let $m \rightarrow \infty$. The convergence of (2.1) is trivial since the necessary and sufficient condition

$$\sum_{i=1}^{\infty} \left(\tau_i - \frac{2k_i(x)}{\cos \phi_i(x)} \right) = \infty$$

is clearly true by $\sum \tau_i = \infty$. κ is continuous except when the trajectory is tangential to the boundary or intersects $\partial Q_i \cap \partial Q_j$ for some i, j .

2.2. The fact that the billiards under consideration are K -flows is shown in [S1] and [BS]. Note that the extra hypothesis of finiteness of the horizon is needed already for ergodicity as shown in [Ke]. In the perturbation result of Section 3 this assumption is not utilized. The Bernoulliness of the flow was proved in [GO] and also follows from the positivity of the Lyapunov exponents [W]. The Bernoulliness will be essential later in α -congruence considerations.

In particular the Ergodic Theorem is applicable to S_t . Since μ is preserved we know that the Lyapunov exponents which as invariant functions are almost everywhere constant satisfy $\lambda_+ + \lambda_- = 0$ μ -a.e. The average expansion rate along a generic orbit, λ_+ , therefore satisfies for μ -a.e. $x \in M$:

$$\lambda_+ = \lim_{t \rightarrow \infty} \frac{1}{t} \int_0^t \kappa(S_s x) ds = \int_M \kappa(x) d\mu(x) \quad (2.4)$$

if $\kappa \in L^1(M)$. The integrability will be verified in Lemma 3.4. Finally by the extension of the Pesin theory to systems with singularities in [KS] which also covers our billiards we know that the measure theoretic entropy of S_t equals to λ_+ .

3. The boundary perturbation theorem

In this section we first state the boundary perturbation result for a canonical dispersive billiard. Our aim is not to find the largest possible subclass of dispersive billiards for which the result is true but rather present methods of proof that can be applied to a large class of tables. The proof that follows is divided into a number of preliminary lemmas. At the end generalizations are briefly discussed.

3.1. Suppose that the dispersive boundary pieces of a c.b.t. are parametrized by arclength r : $\partial\hat{Q} = \{(x_1(r), x_2(r)) \mid r \in I = \bigcup I_i\}$ where I_i are disjoint intervals of $\mathbf{R}$. Let x_i be C^3 functions and $d^{(2)}$ denote the metric defined by the $C^{(2)}$ -norm on this space. The space is complete and separable under this metric since $\{C, \sup\}$ is.

If a function of a billiard flow is continuous under $d^{(2)}$ -small perturbations of $\partial\hat{Q}$ we call it c.u.p. The main result of this section is:

THEOREM 3.2. *The entropy of the billiard flow on a canonical billiard table is continuous under $d^{(2)}$ -small C^3 -perturbations of $\partial\hat{Q}$ that are canonical tables.*

Notice that requiring perturbations to be c.b.t. separately only amounts to requiring the correct corner angles and finiteness of the horizon. The proof of the theorem uses certain machinery that helps us to deal with the function κ . We present that in the form of a few lemmas. The first one establishes a lower bound for the convergence rate of κ_m .

LEMMA 3.3. *Given κ and κ_i as in formula (2.1) we have for all $x \notin M_B$*

$$|\kappa(x) - \kappa_{2m}(x)| \leq \frac{1}{\sum_{i=1}^m \tau_i}.$$

Proof. For the general convergent

$$\frac{P_n}{Q_n} = \frac{1}{b + \frac{1}{b_2 + \frac{1}{\ddots + \frac{1}{b_n}}}}$$

it is known (see e.g. [Kh]) that $Q_n = b_n Q_{n-1} + Q_{n-2}$, $Q_{-1} = 0$, $Q_0 = 1$. Let $P_n/Q_n = \kappa_n$, then $Q_0 = 1$ and $Q_1 = \tau_1$. We claim that $Q_{2m-2} = P_1 + 1$ and $Q_{2m-1} = P_2 + (\tau_1 + \dots + \tau_m)$ imply $Q_{2m} = P_3 + 1$ and $Q_{2m+1} = P_4 + (\tau_1 + \dots + \tau_{m+1})$, where P_i 's are some positive numbers. But this is obvious since

$$b_{2m} = -\frac{k_m}{\cos \phi_m} > 0 \quad \text{and} \quad b_{2m+1} = \tau_{m+1} > 0$$

so induction is complete. Therefore

$$Q_{2m-1} \geq \sum_{i=1}^m \tau_i \quad \text{and} \quad Q_{2m} \geq 1 \quad \forall m \in \mathbf{N}.$$

It also holds that ([Kh])

$$\left| \frac{P_\infty}{Q_\infty} - \frac{P_n}{Q_n} \right| \leq \frac{1}{Q_{n-1} Q_n}$$

when b_i 's are positive, hence the result follows with $n = 2m$. $\square$

Remark. This can also be proved by using the Gauss-Bonnet Theorem as in the derivation of the Riccati equation in the geodesic flow case.

Define the δ -interiors of the table and phase space as

$$Q_\delta = \{q \in Q \mid d(q, \partial \hat{Q}) > \delta\} \quad \text{and} \quad M_\delta = Q_\delta \times S^1$$

for some $\delta > 0$.

LEMMA 3.4. $\kappa \in L^1(M)$ and

$$\int_{M \setminus M_\delta} \kappa(x) d\mu(x) = O(\delta).$$

Proof. We will first prove the formula since it immediately implies the integrability.

Since $\partial \hat{Q}$ is C^3 , $k(q)$ is piecewise continuous on the scatterer and is therefore bounded. Let $q \in Q \setminus Q_\delta$ and suppose that $\partial \hat{Q}_i$ is the nearest component of the scatterer to q . Moreover let $C_q \subset S^1$ be the largest such 'cone' that $x \in \{q\} \times C_q$ implies that $S_{\tau(x)} x \in \partial \hat{Q}_j$, $j \neq i$ (i.e. the ray does not hit the nearest wall first).

First note that for all $x \in (Q \setminus Q_\delta) \times C_q$ by the fact that the raybundle can not focus before next reflection from the scatterer we get that $\kappa(x) < 1/(\tau - \delta)$. Hence

$$\int_{(Q \setminus Q_\delta) \times C_q} \kappa(x) d\mu(x) = O(\delta).$$

To deal with the complement of the cone we use the dispersing relation (2.3). It implies for the incoming raybundle $\kappa_- < 1/\tau - 2k/\cos \phi$ and if $\kappa(x)$ is the curvature of the raybundle between two reflections we have the same bound for this. But then since $d\mu = C \cos \phi dq d\phi$ the singularity is integrable and

$$\int_{(Q \setminus Q_\delta) \times C_q^c} \kappa(x) d\mu(x) = O(\delta).$$

For the integrability we notice that if $x \in M_\delta$ then $\kappa(x) \leq 1/\delta$ and

$$\int_M \kappa(x) d\mu(x) = \int_{M_\delta} + \int_{M \setminus M_\delta} \kappa(x) d\mu(x) \leq \frac{1}{\delta} + O(\delta) < \infty. \quad \square$$

3.5. Next we introduce the notation for the rest of the proof. Given $T > 0$ let

$$M_B^T = \{x \in M \mid S_\tau x \in B \text{ for some } \tau \in [0, T]\}.$$

$(M_B^T)^c$ is open and $\mu(M_B^T) = 0$. Given $x \in M_\delta \cap (M_B^T)^c$ let

$$m(x, T) = \min \left\{ m \in \mathbb{N} \mid \sum_{i=1}^m \tau_i(x) > T \right\}.$$

Notice that $m(x, T) < T/\tau + 1$, $\forall x \in M$. Denote the reflection points by $\{q_i(x)\}$. Call a boundary point δ' -good if its distance along ∂Q to the nearest corner point is at least δ' . Define the δ' - T -tube of x as

$$P_{\delta', T}(x) = \{y \in Q \mid d(y, \pi(S_\tau x)) < \delta' \text{ for some } \tau \in [0, T]\}.$$

The set $P_{\delta', T}(x)$ is *gracing* if $d(\pi(S_\tau x), \partial \hat{Q}_i) < \delta'$ for some $\tau \in [0, T]$, some i and the previous and next (with respect to $S_\tau x$) $q_j(x) \in \partial \hat{Q}_k$, $k \neq i$.

Let $M'_{\delta, T}$ be a collection of $x \in M_\delta \cap (M_B^T)^c$ such that $|\phi_i(x)| < (1 - \delta')\pi/2 \forall i = 1, \dots, m(x, T)$ and that $P_{\delta', T}(x)$ is non-gracing and all the boundary points $\{q_i\}_{i=1}^{m(x, T)}$ are δ' -good. Finally let $\sigma(q)$ denote the direction of the invard normal at $q \in \partial Q$.

LEMMA 3.6. $\kappa_{2m(x)}$ is c.u.p. for all $m \in \mathbb{N}$ and $x \in M'_{\delta, T}$.

Proof. Denote the perturbed boundary by $\partial \tilde{Q}$ and the points of reflection on it by $\tilde{q}_i = \tilde{q}_i(x)$. Clearly if $d^{(2)}(\partial \tilde{Q}, \partial Q) \rightarrow 0$ we have $\|q_i - \tilde{q}_i\| \rightarrow 0$ and $|\sigma(q_i) - \tilde{\sigma}(\tilde{q}_i)| \rightarrow 0 \forall i = 1, \dots, m(x, t)$ for a given $x \in M'_{\delta, T}$. Equivalently for any $\tilde{\delta} \in (0, \delta')$, $\exists \delta''$ such that if $d^{(2)}(\partial \tilde{Q}, \partial Q) < \delta''$ then we have $\tilde{S}_\tau x \in P_{\tilde{\delta}, T} \forall \tau \in [0, T]$. Consequently $d^{(2)}(\partial \tilde{Q}, \partial Q) \rightarrow 0$ implies $\tilde{\tau}_i \rightarrow \tau_i$ and $\tilde{\phi}_i \rightarrow \phi_i$, $i = 1, \dots, m(x, T)$.

If $(x(t), y(t))$, $t \in I$ is a parametric representation of a dispersive curve $\partial \hat{Q}_i$, its curvature is given by

$$\frac{|x'y'' - x''y'|}{[(x')^2 + (y')^2]^{3/2}}$$

which is clearly c.u.p. Therefore $d^{(2)}(\partial \tilde{Q}, \partial Q) \rightarrow 0$ implies $|\tilde{k}(\tilde{q}_i) - k(q_i)| \rightarrow 0 \forall i = 1, \dots, m(x, T)$. Hence τ_i , ϕ_i and k_i are all c.u.p. and since τ_i , $k_i \neq 0$ under small perturbations and $|\phi_i| < (1 - \delta')\pi/2$ for $x \in M'_{\delta, T}$ the result follows. $\square$

Proof of Theorem 3.2. Let the original and perturbed billiard flows be (M, S_t, μ) and $(\tilde{M}, \tilde{S}_t, \tilde{\mu})$. The Lyapunov exponents λ_+ and $\tilde{\lambda}_+$ exist and are finite by Lemma 3.4. Let $D = d^{(2)}(\partial Q, \partial \tilde{Q})$ denote the boundary distance of the tables.

Suppose that $M_{\delta/2} \subset \tilde{M}$. Then

$$\begin{aligned} |\lambda_+ - \tilde{\lambda}_+| &\leq \left| \int_{M_\delta} \kappa(x) \mu(dx) - \int_{M_\delta} \tilde{\kappa}(x) \tilde{\mu}(dx) \right| \\ &\quad + \int_{M \setminus M_\delta} \kappa(x) \mu(dx) + \int_{\tilde{M} \setminus M_\delta} \tilde{\kappa}(x) \tilde{\mu}(dx). \end{aligned}$$

By Lemma 3.4 the third and fourth integral are less than $\varepsilon/3$ for small enough δ and D .

Let the terms on the right-hand side of

$$\left| \int_{M_\delta} \kappa(x) \mu(dx) - \int_{M_\delta} \tilde{\kappa}(x) \tilde{\mu}(dx) \right| \leq \int_{M_\delta} |\kappa(x) - \tilde{\kappa}(x)| \mu(dx) + \sup_{x \in M_\delta} \tilde{\kappa}(x) \|\mu - \tilde{\mu}\|$$

be I and II respectively. Here $\|\cdot\|$ is the total variation on M_δ .

We first analyze II. Since $\tilde{\kappa}(x) \leq \tilde{\kappa}_1(x) = 1/\tilde{\tau}_1$ and $\tilde{\tau}_1 \geq \delta/2 \forall x \in M_\delta$ we see that $\tilde{\kappa}(x)$ is uniformly bounded on M_δ . Moreover since μ and $\tilde{\mu}$ are uniform on M and $\tilde{M}$ we get that $\|\mu - \tilde{\mu}\| \rightarrow 0$ as $D \rightarrow 0$. Hence $\text{II} < \varepsilon/6$ for small enough perturbation.

Let $T = 50/\varepsilon$ and $M'_{\delta,T}$, $P_{\delta',T}$ and $m(x, T)$ be as defined in 3.5. Notice that

$$I \leq \int_{M'_{\delta,T}} |\kappa(x) - \tilde{\kappa}(x)| \mu(dx) + \int_{M_\delta \setminus M'_{\delta,T}} |\kappa(x) - \tilde{\kappa}(x)| \mu(dx).$$

Denote these integrals by III and IV. Clearly

$$IV \leq \sup_{x \in M_\delta \setminus M'_{\delta,T}} |\kappa(x) - \tilde{\kappa}(x)| \mu(M_\delta \setminus M'_{\delta,T}) \leq \frac{2}{\delta} \mu(M_\delta \setminus M'_{\delta,T}) < \frac{\varepsilon}{12}$$

for small δ' by the construction of $M'_{\delta,T}$.

Finally we have to bound the integral III. First we see that

$$|\kappa(x) - \tilde{\kappa}(x)| \leq |\kappa(x) - \kappa_{2m(x,T)}(x)| + |\kappa_{2m(x,T)}(x) - \tilde{\kappa}_{2m(x,T)}(x)| \\ + |\tilde{\kappa}_{2m(x,T)}(x) - \tilde{\kappa}(x)|.$$

By the choice of T and Lemma 3.3 the first term is less than $\varepsilon/50$. Take a sequence of perturbations $\{Q^k\}_{k=1}^\infty$ such that $d^{(2)}(\partial Q^k, \partial Q) \rightarrow 0$ as $k \rightarrow \infty$. Let $\{\kappa_{2m}^k\}$ be the corresponding convergents. Then

$$\lim_{k \rightarrow \infty} \kappa_{2m^k(x,T)}^k(x) = \kappa_{2m(x,T)}(x) \quad \forall x \in M_{\delta',T}$$

since for big enough k $S_\tau^{(k)}x$ is in $P_{\delta'/2,T}(x)$ $\forall \tau \in [0, T]$ (by the Lemma 3.6) and $P_{\delta'/2,T}$ can not be gracing if $d^{(2)}(\partial Q^k, \partial Q) < \delta'/2$. By Egorov's theorem there is $M' \subset M'_{\delta,T}$ such that $\mu(M'_{\delta,T} \setminus M') < \varepsilon\delta/200$ and k_0 such that for $k \geq k_0$ and $\forall x \in M'$

$$|\kappa_{2m^k(x,T)}^k(x) - \kappa_{2m(x,T)}(x)| < \frac{\varepsilon}{100}.$$

Therefore

$$\int_{M'_{\delta,T}} |\kappa_{2m(x,T)}(x) - \kappa_{2m^k(x,T)}^k(x)| \mu(dx) < \frac{2}{\delta} \frac{\varepsilon\delta}{200} + \frac{\varepsilon}{100} = \frac{\varepsilon}{50}.$$

Notice also that $m^k(x, T)$ converges to $m(x, T)$ for μ -a.e. $x \in M'_{\delta,T}$ by the continuity of τ_i^k 's. But again by Egorov's theorem there is $M'' \subset M'_{\delta,T}$ such that $\mu(M'_{\delta,T} \setminus M'') < \varepsilon\delta/200$ and $k_1 \geq k_0$ such that $\forall k \geq k_1, x \in M''$ $|m(x, T) - m^k(x, T)| < 1$. So $\kappa_{2m^k(x,T)}^k(x) = \kappa_{2m(x,T)}(x)$ on M'' and since $|\kappa_{2m^k(x,T)}^k(x) - \kappa^k(x)| < \varepsilon/50$ we also have

$$\int_{M'_{\delta,T}} |\kappa_{2m(x,T)}^k(x) - \kappa^k(x)| \mu(dx) < \frac{2}{\delta} \frac{\varepsilon\delta}{200} + \frac{\varepsilon}{50} = \frac{3\varepsilon}{100}.$$

Consequently $III < \varepsilon/12$.

Since the perturbation space is separable the argument immediately extends to the general case. $\square$

Remarks. (1) The theorem is likely to be true for billiard tables with adjacent dispersive components as well as for tables like the canonical table with α_i angles allowed to be acute. The difficulty in proving it seems purely technical. In Lemma 3.4 the current upperbounds for the curvature fail in the acute case since then essentially $\tau = 0$. Or equivalently κ_3 is not integrable since $\tau_1 + \tau_2$ does not have an absolute lower bound. But it is easy to see that if α is the angle of intersection and $n = \min\{k | \pi/k \leq \alpha\}$ then $\sum_{i=1}^n \tau_i$ has an absolute lower bound. This together

with $\kappa_1 \geq \kappa_3 \geq \kappa_5 \geq \dots \geq \kappa$ in turn implies that we would have to bound κ with either κ_n or κ_{n+1} whichever has odd index. Integrating these for $n \geq 5$ seems quite difficult and provides only little insight into the problem and we omit these extensions. Notice also that $\alpha = 0$ i.e. the existence of a cusp implies the existence of trajectories with $\sum_{i=1}^M \tau_i < \varepsilon$ for any given ε , $M > 0$. Consequently any estimate using κ_n 's would fail.

(2) There is numerical evidence for example in [BSt] that the entropy is continuous for some focusing billiards under boundary perturbations. This can be proved using our approach. Moreover we expect that billiards on 'strictly convex scattering' tables of Wojtkowski ([W]) have continuous entropy under C^4 -perturbations of the boundary (these smooth perturbations ensure that the mixing properties that are needed later will not change).

4. α -congruence

Although the continuity of the entropy under perturbations is an interesting result *per se* we will see that it leads to a considerably stronger statement once it is coupled with certain results from the Bernoulli theory. Showing this is the main purpose of the present section. We proceed in this by first defining the concept of α -congruence and stating the main theorem. After the proof we discuss briefly some implications of the result. The definition is as in [OW] which we refer to for further motivation and interpretations.

Definition 4.1. Two measure-preserving flows (M, f_t, μ) and $(M, \tilde{f}_t, \tilde{\mu})$ on a compact metric space (M, d) are α -congruent if there is an isomorphism ι (i.e. an invertible measure-preserving transformation such that $\iota \circ f_t = \tilde{f}_t \circ \iota$) and ι moves all but α of the points of M by less than α i.e. $\mu(\{x \in M \mid d(\iota(x), x) \geq \alpha\}) < \alpha$.

Remark. (1) If the flows are also ergodic then by the Ergodic Theorem

$$\frac{1}{T} \int_0^T d((\tilde{f}_t \circ \iota)(x), f_t(x)) dt \rightarrow \int_M d(\iota(x), x) d\mu \quad \mu\text{-a.e.}$$

i.e. the definition is equivalent to requiring that (in addition to the isomorphism) the infinite trajectories of f_t and $\tilde{f}_t$ are almost surely within α of each other for all except density α of times. Hence in verifying α -congruence between two flows it suffices to determine their $\bar{d}$ -distance for α -fine partitions as defined in the Appendix.

(2) The concept has a natural interpretation in terms of a viewer. If a viewer is observing the space M with resolution α and commits an experimental error with probability α (density of times that the observations are not correct) then f_t and $\tilde{f}_t$ are indistinguishable to the viewer.

(3) α -congruence is an attempt to remedy certain shortcomings of the concept of structural stability. In the latter the flows are conjugated by a homeomorphism which however in general preserves none of the measure-theoretic structure of the systems (typically a set of trajectories of full measure gets mapped onto a set of measure zero). By giving up the continuity in the conjugation and allowing the map to fail to couple on a small set we obtain a concept that seems more natural for the purposes of smooth dynamics and moreover has greater versatility as shown in [OW] and [EI].

(4) For the case where the flows act on different spaces the Definition 4.1 needs to be modified. However in the case of the billiard at hand the change is trivial since the phase spaces are embedded in the same space ($\mathbb{R}^3$) and inherit the metric from it. For simplicity we will from now on assume that this metric has its diameter bounded by one.

The main result is:

THEOREM 4.2. *Given $\alpha > 0$ the billiard flow (M, S_t, μ) on a canonical table is α -congruent with a sufficiently small time scaled perturbation $(\tilde{M}, \tilde{S}_t, \tilde{\mu})$ of the type described in Theorem 3.2. Here $c \rightarrow 1$ as $\alpha \downarrow 0$.*

In proving this it is instrumental that the processes are Bernoulli and hence finitely determined. For the definition of this as well as certain concepts used in the subsequent proofs see the Appendix.

Definition. By a *regular partition* of the phase space M of the canonical billiard we mean a finite partition $\mathcal{P} = \{P_i\}_{i=1}^N$ where $P_i = C_i \cap M$ and $\{C_i\}_{i=1}^N$ is a disjoint cover of M with cubes of sidelength δ_N . If we have a collection of manifolds $\{M^n\}$ we let $M = \bigcup_n M^n$ and define regular partition for this as above. Then the *induced partitions* are formed by $P_i^n = C_i \cap M^n$.

LEMMA 4.3. *Let (M, S_t, μ) be a canonical billiard system $\{(M^n, S_t^n, \mu^n)\}$ a sequence of boundary perturbed billiards converging to it. Let $\mathcal{P} = \{P_i\}_{i=1}^N$ be a regular partition and $\{\mathcal{P}^n\}$ the induced partitions. Then*

$$|h(S_{t_0}, \mathcal{P}) - h(S_{t_0}^n, \mathcal{P}^n)| \rightarrow 0.$$

Proof. Since fine partitions are not necessary generating we cannot conclude this from Theorem 3.2. Due to the singularities in the fibres the argument here is necessarily a little more complicated than in the geodesic flow case (see [OW]). The idea is to show that the size of a typical future N - $\mathcal{P}$ -atoms does not change much in small perturbations. We only need to consider the equipartitioning property on a small test set by the local Shannon-McMillan-Breiman Theorem.

For a.e. $x \in M$ we have both the contracting and expanding fibres through it. Denote these by $\gamma^c(x)$ and $\gamma^e(x)$. They have positive length and by their construction (in [S1]) given $\varepsilon > 0$ we can define a test box $Q_\varepsilon \supset M$ as follows. Let Q_ε be such that $\delta(\varepsilon) = \mu(Q_\varepsilon)$, $\text{diam}(Q_\varepsilon) \leq \delta(\varepsilon)$ and it is *well fibred* in the sense that most of its points are on 'long' smooth fibres i.e. the fibres do not have cusps in Q_ε . Defining $\gamma_\varepsilon^c(x)$ to be the connected smooth component of $\gamma^c(x) \cap Q_\varepsilon$ containing x and similarly for the expanding fibre we formally require that

$$\frac{\mu(\{x \in Q_\varepsilon \mid \gamma_\varepsilon^c(x) \text{ and } \gamma_\varepsilon^e(x) \text{ exist}\})}{\mu(Q_\varepsilon)} \geq 1 - \frac{\varepsilon}{2}.$$

Obviously $\delta(\varepsilon) \downarrow 0$ as $\varepsilon \downarrow 0$. Moreover we assume that ∂Q_ε is nice in the sense that

$$\mu(\{x \in M \mid d(x, \partial Q_\varepsilon) \leq \Delta\}) = O(\Delta).$$

The existence of such test box follows from the existence of the fibrations and their absolute continuity.

Let $\mathcal{P}^N = \bigvee_{i=1}^N T^i \mathcal{P}$ where $T = S_{t_0}$ for some $t_0 > 0$. Choose N such that the $\mathcal{P}^N$ -atoms for which there is a constant C such that

$$C e^{-N(h(T, \mathcal{P}) + \varepsilon)} \leq \frac{\mu(P_i^N \cap Q_\varepsilon)}{\mu(Q_\varepsilon)} \leq C e^{-N(h(T, \mathcal{P}) - \varepsilon)}$$

fill at least a set of measure $(1 - \varepsilon/2)\mu(Q_\varepsilon)$ in Q_ε . This is possible by the SMB Theorem.

Let S_ε be an arbitrary well-fibred two-dimensional section of Q_ε spanned by the flow direction and the contracting fibres. The well fibering is in the aforementioned sense only the measure being restricted to the section S_ε . It is covered by $\bigvee_{i=0}^\infty T^{-i} \mathcal{P}^N$ -atoms and by the absolute continuity of the expanding fibres an atom of correct size (in the sense of the SMB Theorem) in $\mathcal{P}^N$ -partition induces (through the $\bigvee_0^\infty$ -joining) such in the section. Equivalently the number of these generic atoms covering all but fraction ε of the Q_ε -box is the same as that covering the section and growing at the exponential rate

$$h(T, \mathcal{P}) \stackrel{\text{def}}{=} H\left(\mathcal{P} / \bigvee_{i=1}^\infty T^{-i} \mathcal{P}\right) = H\left(\mathcal{P}^N / \bigvee_{i=1}^\infty T^{-i} \mathcal{P}^N\right).$$

Given $\gamma_\varepsilon^c(x)$ we now claim that for most $x \in S_\varepsilon$ it is covered by almost the same number of atoms as S_ε .

Consider $\tilde{\gamma} \subset \gamma_\varepsilon^c(x)$ which is a segment with constant N - $\mathcal{P}$ -name. It is sufficient to show that its continuous image under S_t in Q_ε (a strip in S_ε) intersects only with relatively few $\mathcal{P}^N$ -atoms. This is because given any $x \in \tilde{\gamma}$ the number of different N - $\mathcal{P}$ -names of $S_t x$, $t \in (0, \delta)$ is proportional to N . This in turn is implied by the fact that a name can change only through $\mathcal{P}$ -boundary crossing of at least one of $T^i x$, $i = 0, \dots, N-1$ and the boundary layer is thin by our assumption on the niceness if the partition $\mathcal{P}$. Since the diameter of Q_ε is bounded by $\delta(\varepsilon)$ the argument applies to any flow translate of $x \in S_\varepsilon$ and the number of atoms thus obtained along the strip is $o(e^N)$. Hence for most $x \in S_\varepsilon \gamma_\varepsilon^c(x)$ is covered by $e^{N\tilde{h}}$ N - $\mathcal{P}$ -atoms, $\tilde{h} \in (h(T, \mathcal{P}) - C\varepsilon, h(T, \mathcal{P}))$ for some C independent of ε .

By our earlier results we know that the contracting fibres of (M^n, S_t^n, μ^n) converge to those of (M, s_t, μ) . Hence for large enough n the fibres are covered by the same number of N - $\mathcal{P}$ -atoms. Hence the set Q_ε is covered by almost the same number of atoms and the partition entropies are close. $\square$

Proof of Theorem 4.2. Fix ε , $t_0 > 0$. Given a ε -fine partition $\mathcal{P}$ ($\text{diam}(P_i) < \varepsilon$) and induced partitions $\{\mathcal{P}^n\}$ we have by the previous lemma that

$$|h(S_{t_0}, \mathcal{P}) - h(S_{t_0}^n, \mathcal{P}^n)| < \delta \quad \forall n \geq n_0.$$

The correct δ to be used will be specified later. By choosing an even smaller perturbation we can match the finite distributions of $(S_{t_0}, \mathcal{P})$ and $(S_{t_0}^n, \mathcal{P}^n)$. Note that although M and M^n are different $M \Delta M^n$ is negligible and the distribution matching can be done. By the fact that the flows are Bernoulli and hence finitely determined we can choose δ in the beginning in such a way that $\bar{d}((S_{t_0}, \mathcal{P}), (S_{t_0}^n, \mathcal{P}^n)) < \varepsilon/3$. Hence for the continuous time distance we have $\bar{d}((S_t, \mathcal{P}), (S_t^n, \mathcal{P}^n)) < 2\varepsilon/3$. From Theorem 3.2 we conclude that $h(S_1) = h(S_{c_n}^n)$ for

some sequence $c_n \rightarrow 1$. Therefore by the Isomorphism Theorem S_t and $S_{c_n}^n$ are isomorphic Bernoulli flows. $\bar{d}((S_t^n, \mathcal{P}^n), (S_{c_n}^n, \mathcal{P}^n)) \rightarrow 0$ as $n \rightarrow \infty$ (by a continuous time analog of Theorem 3 p 41 in [O]) and we see that the $\bar{d}((S_t, \mathcal{P}), (S_{c_n}^n, \mathcal{P}^n)) < \varepsilon$. Finally since the partition $\mathcal{P}$ is ε -fine S_t and $S_{c_n}^n$ are ε -close in the $\bar{d}$ -metric that is defined using the original metric d of the space. Hence by Remark (1) following Definition 4.1 the flows are α -congruent with $\alpha = 2\varepsilon$. $\square$

Remarks 4.4. (1) α -congruence can only be shown for billiards that are Bernoulli flows. If one of the flows is Bernoulli and the other is of zero entropy (for example just ergodic) the entropies obviously cannot be matched. If the other is a non-Bernoulli flow with positive entropy (for example a K -flow) then the Bernoulli flow can only be isomorphic to its Bernoulli factor. For a more detailed account of this see [OW].

(2) The concept of a viewer offers an interesting interpretation of these results. Suppose that the toral table corresponding to the one in figure 1(b) is perturbed in such a way that the disk becomes a convex polygon. If the boundaries are closer than ε in d -metric then an ε -good viewer cannot distinguish the tables. But in the case of a polygon the flow is at most ergodic. Hence the viewer can observe the non- ε -congruence and deduce the existence of the 'invisible' boundary perturbation from this.

5. The smooth billiard

The objective of this section is to consider a geodesic flow on a manifold that approximates a dispersive billiard table. First observations of this flow were made by Arnold (see e.g. [AA]). We will show that its entropy is close to that of a certain billiard and that it can be considered as a statistically stable perturbation of the latter. This serves as an example of the robustness of the concept of α -congruence and we will see that the perturbation is not structurally stable even in a wide sense.

5.1. Our smooth billiard is the geodesic flow on the unit tangent bundle of a smooth surface $Q \subset \mathbb{R}^3$ illustrated in figure 2. The "table" consists of two flat tori (unit squares) that are joined by two necks of total sectional (Gauss) curvature -8π (genus 3). We partition $Q = Q_F^+ \cup Q_C \cup Q_F^-$ where F and C refer to the flat and curved parts and the signs to top and bottom sheets (M is split analogously). The arrows indicate the (y, z) -section that is shown on the right.

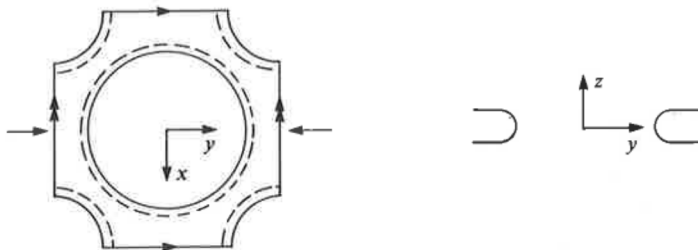


FIGURE 2

It is necessary that the necks are surfaces of revolution of smooth convex (in (y, z) -coordinates) curves. As will become obvious later our arguments do not depend in any essential way on the curve and therefore for technical simplicity we will assume that the necks are half-tori. Hence the curvatures of the necks are

$$K_i^r = \frac{\cos \theta}{r[R_i + r(1 + \cos \theta)]} \quad \theta \in [\pi/2, 3\pi/2]. \quad (5.1)$$

Whenever there is no danger of confusion we will suppress the index r (radius of the circular section) in the notation.

Definition 5.2. The geodesic flow G_t on Q acts on the unit tangent bundle $M = SQ$ by translating a unit vector x along a geodesic γ_x to the vector $\dot{\gamma}_x$. We assume that R_i and r are such that the flow has finite horizon i.e. the visits to the necks occur at bounded intervals for all x . This flow is called the *standard smooth dispersive billiard* or in short SSDB.

The geodesic flow preserves the Liouville measure $d\mu = d\sigma d\omega$ where $d\sigma$ is the area element induced by the metric and $d\omega$ is the Lebesgue measure on S^1 .

The *standard dispersive billiard* (SDB) which we denote by (M, S_r, μ) is defined on the table obtained from the table in figure 1(b) by threefold reflection. Then the opposite flat boundary pieces are identified, i.e. the table is a flat torus with two circular obstacles. These are of diameter R_i therefore the flow has finite horizon. Note that the total curvature of the boundary and hence the manifold M is -8π . Our aim is to consider the convergence of the sequence $\{(M^r, G_t^r, \mu^r)\}_{r>0}$ to (M, S_r, μ) as r approaches zero. We start by introducing some more notation and describing the basic properties of toral geodesics.

Let $\hat{Q}^r(\hat{Q})$ denote the universal cover of $Q^r(Q)$ i.e. the periodic extension of $Q^r(Q)$ with respect to the flat boundaries in figures 2 and 1 (the construction naturally extends to $M^r(M)$ as well). Define d^C to be the length of the longest straight line on $\hat{Q}$ and for a fixed small $r_0 > 0$ let

$$d_C = \inf_{0 < r < r_0} \{\text{shortest distance between two necks along } M_F^r\}.$$

Let $\pi^r: Q^r \rightarrow Q$ be the projection $(x, y, z) \mapsto (x, y)$. We use the same symbol for the projection from M^r onto M as well. Hence in particular $\pi^r(M^r) = M$. The projection $\pi: SQ^r \rightarrow Q^r$ is as before. In the following we occasionally consider the curvature K^r as a function on SQ^r .

PROPOSITION 5.3. Let T be the surface of revolution obtained by revolving a circle $(y - R - r)^2 + z^2 = r^2$ around the z -axis. Let γ_x be a geodesic on T , $\alpha(t)$ the angle between it and the meridian through $G_t x$ and $r(t)$ the distance of $G_t x$ from the z -axis. Then we have Clairaut's relation:

$$r(t) \sin \alpha(t) = C.$$

The proof of this can be found in standard references for differential geometry e.g. [Sp]. By applying Clairaut's relation we can further characterize the geodesics. We now assume that $\alpha(0)$ opens inwards and is not blunt.

LEMMA 5.4. Let p_+ , p_- and p_0 be the top, bottom and inside parallels of T . Suppose that $\pi(x) \in p_{\pm}$ and let $\tau = \inf \{t > 0 \mid \alpha(t) = \pi/2 \text{ or } \pi(G_t x) \in p_0\}$. Then

(i) $\exists! \alpha_c \in (0, \pi/2)$ such that for $\alpha(0) = \alpha_c$ $\tau = \infty$. For $\alpha(0) \neq \alpha_c$ τ is finite and either $0 \leq \alpha(0) < \alpha_c : \pi(x) \in p_{\pm} \Rightarrow \pi(G_{2\tau} x) \in p_{\mp}$ and $\alpha(2\tau) = \alpha(0)$ or

$$\alpha_c < \alpha(0) \leq \pi/2 : \pi(x) \in p_{\pm} \Rightarrow \pi(G_{2\tau} x) \in p_{\pm}$$

and $\alpha(2\tau) = \pi - \alpha(0)$.

(ii) $\lim_{r \downarrow 0} \alpha_c = \pi/2$.

(iii) $\forall \delta > 0$ $\pi/2 - \delta < \alpha(0) < \pi/2$ implies that $|\alpha(t) - \pi/2| < \delta \forall t \in [0, 2\tau]$.

Proof. Let $\pi(\tilde{x}) \in p_{\pm}$ and $G_t \tilde{x}$ be the geodesic that satisfies $R = (R+r) \sin \alpha(0)$. For $\alpha(\tau) = \pi/2$ the tangent vectors to $G_t \tilde{x}$ and p_0 at $\pi(G_t \tilde{x})$ agree hence $G_t \tilde{x} \in p_0$ for $t \geq \tau$ by the uniqueness of the geodesics on a smooth surface. But this is impossible unless $\tau = \infty$ (i.e. $G_t \tilde{x}$ spirals in). Define $\alpha_c = \arcsin(R/(R+r))$ and (ii) follows.

If $0 \leq \alpha(0) < \alpha_c$ then by Clairaut's formula $R > (R+r) \sin \alpha(0)$ so for some $\varepsilon(\alpha(0)) > 0$

$$0 \leq r(t) \sin \alpha(t) \leq R - \varepsilon.$$

Since $r(t) \geq R$ we get

$$0 \leq \alpha(t) \frac{\pi}{2} - \delta(\alpha(0)) \quad \forall t \text{ and for some } \delta > 0.$$

So $\tau < \infty$ and $\pi(x) \in p_{+} \Rightarrow \pi(G_{2\tau} x) \in p_{-}$ by the symmetry with respect to p_0 .

If $\alpha_c < \alpha(0) < \pi/2$ we have $R < (R+r) \sin(0)$ hence

$$R + \varepsilon \leq r(t) \quad \text{for some } \varepsilon(\alpha(0)) > 0.$$

Suppose that $\alpha(t) < \pi/2 \forall t$ i.e. $G_t x$ spirals in. Then the ω -limit set of $\{G_t x\}_{t \geq 0}$ would be a parallel and geodesic. But there are none between $p_{\pm}$ and p_0 . Hence for some finite $\tau \alpha(\tau) = \pi/2$ and we get $\pi(x) \in p_{+} \Rightarrow \pi(G_{2\tau} x) \in p_{+}$.

For $\alpha(0) \neq \alpha_c$ by Clairaut's relation and symmetry $\alpha(2\tau) = \alpha(0)$ or $\pi - \alpha(0)$ depending whether the geodesic changes sheets or not. This completes (i). Finally the inequality $\sin \alpha(0) \leq \sin \alpha(t)$ implies (iii). $\square$

6. Ergodic properties and the second perturbation theorem

The ergodic properties of a geodesic flow in the non-uniformly hyperbolic case are investigated in detail in [P]. In particular it is shown that

THEOREM 6.1. A geodesic flow on a two-dimensional compact manifold of genus greater than one and without focal points is isomorphic to a Bernoulli flow.

By [Eb] the existence of focal points implies positive curvature somewhere on the manifold hence the SSDB G_t is isomorphic to a Bernoulli flow. In [P] the Pesin formula $h_{\mu} = \lambda_{+}$ for the geodesic flow in the case at hand is established. h_{μ} is the measure theoretic entropy of the flow and λ_{+} is its positive Lyapunov exponent. In order to determine the Lyapunov exponents of the flow one needs to examine the fibration of the manifold into contracting and expanding submanifolds. This has been done for example in [AA] and [S2]. By these classical results the exponent

of the contraction rate along the fibres equals to the curvature of the fibre (or the horocycle). Hence the Lyapunov exponents are obtained by averaging this curvature along generic trajectories as shown in § 2.2.

The main result of this section is:

THEOREM 6.2. *Let (M, S_t, μ) be the SDB and $\{(M_r, G_t^r, \mu^r)\}_{r>0}$ the SSDBs of sections 5.1 and 5.2. Then $\lambda_+^r \rightarrow \lambda_+$.*

The analysis of the contracting fibres that determine the Lyapunov exponents is rather technical. We proceed to examine them in a series of lemmas describing the evolution of the curvature of a geodesic section that culminate to the proof the above. We will omit the index r until the actual convergence results appear.

LEMMA 6.3. *Let $k(G_t x)$ denote the absolute value of the curvature of a section of an infinitesimal geodesic ray bundle around $G_t x$. Then*

$$k(G_{t_2} x) - k(G_{t_1} x) = \int_{t_1}^{t_2} K(G_t x) dt + \int_{t_1}^{t_2} k^2(G_t x) dt. \quad (6.4)$$

Proof. This is just the Riccati equation integrated. See e.g. [M].

Let $\gamma(x)$ denote the local contracting fibre through x and $\kappa(x)$ the absolute value of its curvature. These exist for μ -a.e. $x \in M$. Let $\tilde{\gamma}(x)$ denote a smooth approximate fibre and $\tilde{\kappa}$ the corresponding fibre curvature. $\tilde{\gamma}(x)$ is an orthogonal section of geodesic rays at x and $\tilde{\kappa}(G_t x)$ is either 0 or ∞ for some finite t .

LEMMA 6.4. *If $\tilde{\kappa}(G_t x) \neq 0, \infty \forall t \in [0, T)$ then $|\kappa(x) - \tilde{\kappa}(x)| \leq 1/T$.*

Proof. Let $\underline{\kappa}(x)$ and $\bar{\kappa}(x)$ be the curvatures of the approximative fibres that yield parallel and focusing sections respectively at time T . Obviously $\underline{\kappa}(x) \leq \kappa(x) \leq \bar{\kappa}(x)$. If we denote $\Delta(t) = \bar{\kappa}(G_t x) - \underline{\kappa}(G_t x)$ then by the previous lemma $\dot{\Delta} = (\bar{\kappa} + \underline{\kappa})\Delta$. But the solution of this equation is minored by the solution of $\dot{\Delta} = \Delta^2$ which blows up before time T if $\Delta(0) > 1/T$. Hence $(\bar{\kappa} - \underline{\kappa})(0) \leq 1/T$ as claimed. $\square$

If $x \in \partial M_C$ is such that its direction satisfies $\alpha(0) \in (-\pi/2, -\pi/2 + \delta) \cup (\pi/2 - \delta, \pi/2)$ we call it δ -oblique. If $z \in M_C$ and $z = G_t x$, $x \in \partial M_C$, $t > 0$ and $G_s x \in M_C \forall s \in [0, t]$ we say that z is δ -oblique if x is. This is consistent with Lemma 5.4(iii). Also define

$$B_{\delta, n} = \{x \in M \mid \text{the } n\text{th arrival of } G_t x \text{ to } M_C \text{ is } \delta\text{-oblique}\}.$$

We now proceed to establish two lemmas that will enable us to ignore oblique trajectories in the main argument.

LEMMA 6.5. *There is $C(n)$ such that*

$$\int_{B_{\delta, n}} \kappa(x) \mu(dx) < C(n) \delta.$$

For a family $\{(M^r, G_t^r, \mu^r)\}_{r>0}$ this estimate holds uniformly i.e. C is independent of r .

Proof. Let $x \in M$ and define $\{\alpha_i\}_{i=1}^\infty$ and $\{\omega_i\}_{i=1}^\infty$ be the times when $G_t x$ enters and leaves M_C . If $G_t x$ is gracing M_C we have $\alpha_i = \omega_i$ or $\alpha_i = \omega_{i+1}$ depending on whether

$x \in M_F$ or $x \in M_C$. To avoid complicating the notation and without loss of generality we assume in sequel the former. To prevent focusing on M_F we need

$$1 - d_C \kappa(G_{\omega_i} x) > 0$$

hence $\kappa(G_{\omega_i} x) < 1/d_C \forall x \in M$. This bound together with Lemma 6.3 implies

$$\int_{\omega_i}^{\omega_{i+1}} \kappa^2(G_i x) dt < \frac{1}{d_C} - \int_{\omega_i}^{\omega_{i+1}} K(G_i x) dt.$$

For a fixed $x \in M$ let

$$I(L) = \bigcup_{i=1}^{\infty} \{[\omega_i, \omega_{i+1}] \cap [0, L] \mid G_{\alpha_{i+n}} x \text{ is } \delta\text{-oblique}\}$$

and let $m(L)$ be the number of intervals in $I(L)$. Then

$$\frac{1}{L} \int_{I(L)} \kappa^2(G_i x) dt \leq \frac{m(L)}{d_C L} - \frac{1}{L} \int_{I(L)} K(G_i x) dt. \quad (6.5.1)$$

By the Ergodic Theorem

$$\frac{1}{L} \int_{I(L)} \kappa^2(G_i x) dt = \frac{1}{L} \int_0^L f_{\delta}(G_i x) dt \rightarrow \int_M f_{\delta}(z) \mu(dz) \quad (6.5.2)$$

for μ -a.e. $x \in M$ where

$$f_{\delta}(z) = \begin{cases} \kappa^2(z) & \text{if the } n\text{th arrival of } G_i z \text{ to } M_C \\ & \text{is } \delta\text{-oblique} \\ 0 & \text{otherwise.} \end{cases}$$

Hence

$$\int_M f_{\delta}(z) \mu(dz) = \int_{B_{\delta,n}} \kappa^2(z) \mu(dz). \quad (6.5.3)$$

With similar argument we get

$$\frac{1}{L} \int_{I(L)} K(G_i x) dt \rightarrow \int_{B_{\delta,n}} K(z) \mu(dz) \quad \mu\text{-a.s.}$$

Let

$$B_{\delta,n}(q) = \{U \subset S^1 \mid \pi(x) = q, x \in Q \\ \times U \text{ and the } n\text{th arrival of } G_i x \text{ to } M_C \text{ is } \delta\text{-oblique}\}.$$

Consequently $B_{\delta,n}$ is skew product of Q and $B_{\delta,n}(\cdot)$. Let

$$P = \sup_{q \in Q} \{\text{number of necks visible from } q \text{ on } \hat{Q}\}.$$

Since G_i^r have finite horizon P is finite and independent of r . The number of obstacles reachable at the n th reflection is bounded by P^n (with the possibility of the same obstacle counted again at different reflections). Since the obstacles are convex we see that

$$\sup_{q \in Q} \omega(B_{\delta,n}(q)) \leq 2P^n \sup_{q \in Q} \omega(\{\text{largest connected bundle of geodesics from } q$$

that are δ -oblique at the n th bounce}).

But the supremum is clearly bounded by some $C\delta$, C independent of r (in fact considering the case of gracing a circle it is easy to establish the bound

$C \min \{\delta, \delta^2/d\}$ where d is the length of the geodesic). Therefore

$$\begin{aligned} - \int_{B_{\delta,n}} K(z) \mu(dz) &= - \int_Q K(q) \int_{B_{\delta,n}(q)} d\omega \sigma(dq) \\ &\leq - \int_Q K(q) \sigma(dq) P^{n-1} \sup_{q \in \partial M_N} \omega(B_{\delta,1}(q)) \\ &\leq 16\pi C P^n \delta. \end{aligned} \quad (6.5.5)$$

Clearly

$$\lim_{L \rightarrow \infty} \frac{m(L)}{d_C L} \leq C \mu(B_{\delta,n}) \leq C \delta. \quad (6.5.6)$$

Formulas (6.5.1)–(6.5.6) imply

$$\int_{B_{\delta,n}} \kappa^2(x) \mu(dx) \leq C(n) \delta.$$

Finally

$$\int_{B_{\delta,n}} \kappa(x) \mu(dx) \leq \int_{B_{\delta,n} \cap \{x | \kappa(x) \leq 1\}} \mu(dx) + \int_{B_{\delta,n} \cap \{x | \kappa(x) > 1\}} \kappa^2(x) \mu(dx) \leq C \delta.$$

Since the total curvature of Q_C is independent of r the estimate is true for all $r \in (0, r_0)$ some $r_0 > 0$. $\square$

Define $B_\delta(T) = \{x \in M \mid \text{for some } t \in [0, T] G_t x \text{ is } \delta\text{-oblique}\}$.

LEMMA 6.6. For given ε , $T > 0$ we can find $\delta > 0$ such that

$$\int_{B_\delta(T)} \kappa(x) \mu(dx) < \varepsilon.$$

This is again uniform in r .

Proof. Recall that $B_{\delta,n} = \{x \in M \mid G_{\alpha_n} x \text{ is } \delta\text{-oblique}\}$. So $B_\delta(T) \subset \bigcup_{n=1}^N B_{\delta,n}$ where $N = [T/d_C]$. But then by the previous lemma

$$\int_{B_\delta(T)} \kappa(x) \mu(dx) \leq \sum_{n=1}^N \int_{B_{\delta,n}} \kappa(x) \mu(dx) < C(N) \delta.$$

Since $N(r)$ is uniformly bounded the result follows. $\square$

Remark. By the uniformity in r in Lemmas 6.5 and 6.6 we obtain the bounds to the SDB as well.

Let $M'_C(\delta) = \{x \in M'_C \mid x \text{ is not } \delta\text{-oblique}\}$. Analogously to Lemma 3.4 we have

LEMMA 6.7. Given δ , $\varepsilon > 0 \exists r_0 > 0$ such that

$$\int_{M'_C(\delta)} \kappa^r(x) \mu^r(dx) < \varepsilon \quad \forall r \in (0, r_0).$$

Proof. If $K^r = 1/(Rr)$, then for toral necks $K^r \geq -K^r(x) \forall x \in M^r$. Let $x \in M'_C(\delta)$. Since formula (6.4) holds for κ^r we see that if $\kappa^r(x) \geq \sqrt{K^r}$ then $\kappa^r(G_t x) - \kappa^r(x) \geq 0$ for $t \geq 0$ as $G_t x \in M'_C$. Let $y = G_s x \in M'_C(\delta)$ be the exit point from M'_C . It exists for small enough r , precisely as soon as $\alpha'_c > \pi/2 - \delta$. Then

$$\sqrt{K^r} \leq \kappa^r(x) \leq \kappa^r(G_t x) \leq \kappa^r(y) \leq \frac{1}{d_C} \quad \forall t \in [0, s].$$

But $K^r \rightarrow \infty$ as $r \downarrow 0$ hence the inequalities are violated and $\kappa^r(x) < \sqrt{K^r} \forall x \in M_C^r(\delta)$. Therefore

$$\int_{M_C^r(\delta)} \kappa^r(x) \mu^r(dx) \leq \frac{C}{\sqrt{r}} \int_{M_C^r} \mu^r(dx) = O(\sqrt{r}). \quad \square$$

The dispersion relation is now approximate.

LEMMA 6.8. Let $\delta > 0$, $x \in \partial M_C^r$, $\alpha(0) \in (-\pi/2 + \delta, \pi/2 - \delta)$ and τ be as in Lemma 5.4. Then

$$k(G_{2\tau}^r x) - k(x) = -\frac{2}{R_i \cos \alpha(0)} (1 + O(r)).$$

The correction is independent of x .

Proof. By Lemma 6.3, some elementary geometry and formula (5.1) we have

$$\begin{aligned} k(x) &= k(G_{2\tau}^r x) - \int_0^{2\tau} K^r(G_t^r x) dt - \int_0^{2\tau} [k(G_t^r x)]^2 dt \\ &\leq \frac{1}{d_C} - \int_0^{2\tau} K^r(G_t^r x) dt = \frac{1}{d_C} + \frac{2}{R_i \cos \alpha(0)} (1 + O(r)). \end{aligned}$$

Since $\lim_{r \downarrow 0} K^r(y) = -\infty$ for all y in the interior of M_C we see that $k(G_t^r x)$ is decreasing for $t \in [0, 2\tau]$ for small enough r . But then by Lemma 6.3 again

$$\begin{aligned} k(G_{2\tau}^r x) &= k(x) + \int_0^{2\tau} K^r(G_t^r x) dt + O(r) \\ &= k(x) - \frac{2}{R_i \cos \alpha(0)} (1 + O(r)). \quad \square \end{aligned}$$

Recall that the r -interior of the phase space of SDB is denoted by $M_r = \{x \in M \mid d(x, \partial M) \geq r\}$. Define

$$M_r(\delta, T) = \{x \in M \mid d(x, \partial M) \geq r, S_r x \text{ is not } \delta\text{-oblique for } t \in [0, T]\}.$$

The corresponding set for SSDB is of course $M_F^r(\delta, T) = M_F^r \cap (B_\delta(T))^c$.

The following result is a direct consequence of Lemma 6.8 and the relation (2.2).

LEMMA 6.9. Let $\{\tau_i^r\}$ be the lengths of the intervals that $G_t^r x$ spends on M_F^r and let $\{\phi_i^r\}$ be the angles (measured from the meridians) at which it enters M_C^r . Then for $x \in M_F^r(\delta, T)$ and $n \leq (T-1)/d^C$

$$\begin{aligned} \kappa_{2n}^r(x) &= \frac{1}{\tau_1^r + \frac{1}{\frac{2}{R_{i_1} \cos \phi_1^r} (1 + O(r)) + \frac{1}{\tau_2^r + \frac{1}{\ddots + \frac{1}{\frac{2}{R_{i_n} \cos \phi_n^r} (1 + O(r))}}}} \end{aligned}$$

is the curvature of a infinitesimal geodesic ray bundle along $G_t^r x$ which is parallel after n th exit from M_N .

On the set of points generating non-oblique trajectories on $[0, T]$ we have a natural convergence result.

LEMMA 6.10. Let $\kappa_{2n}(x)$ be as defined in § 2.1 for the SDB (M, S, μ) . Then for any $\delta, T > 0$ and $n \leq (T-1)/d^C$ we have

$$\kappa_{2n}^r(x) - \kappa_{2n}(\pi^r(x)) \rightarrow 0 \quad \text{as } r \downarrow 0$$

uniformly on $M'_F(\delta, T)$ (and hence $\pi^r(x) \in M_r(\delta/2, T)$ for small r).

Proof. Let $x \in M'_F(\delta, T)$. First notice that $\tau_i^r \leq d^C$ and $n \leq (T-1)/d^C$ imply

$$\sum_{i=1}^n \tau_i^r \leq T-1.$$

Let α_i and ω_i be as in the proof of Lemma 6.5, then

$$\omega_n^r = \sum_{i=1}^n \tau_i^r + \sum_{i=1}^n (\omega_i^r - \alpha_i^r)$$

and since $G_r^r x$ is not oblique on $[0, T]$, $\omega_i^r - \alpha_i^r \rightarrow 0$ as $r \downarrow 0$ for $\alpha_i^r \in [0, T]$. Therefore $\omega_n^r < T$ for small r . Hence $\kappa_{2n}^r(x)$ and $\kappa_{2n}(\pi^r(x))$ are the curvatures of focusing ray bundles that are parallel at the latest at T and are not δ -oblique. Clearly $\tau_i^r(x) \rightarrow \tau_i(\pi^r(x))$ and $\phi_i^r(x) \rightarrow \phi_i(\pi^r(x))$ uniformly in x for $i = 1, \dots, n$. Hence Lemma 6.9 together with the uniformity in x established in Lemma 6.8 imply the result.

Proof of Theorem 6.2. Given $\varepsilon > 0$ choose $T_c \geq 18/\varepsilon$. Let $n = \lfloor T_c/d^C \rfloor$ and take T_0 such that $T_0 > nd^C + 1$. Furthermore let $M^r(\delta, T_0) = M^r \cap (B_\delta(T_0))^C$ and

$$M(\delta, T_0) = \{x \in M \mid S_r x \text{ is not } \delta\text{-oblique on } [0, T_0]\}.$$

Then by Lemma 6.6 we have

$$\left| \int_{M^r} \kappa^r d\mu^r - \int_M \kappa d\mu \right| \leq \left| \int_{M^r(\delta, T_0)} \kappa^r d\mu^r - \int_{M(\delta, T_0)} \kappa d\mu \right| + \frac{\varepsilon}{6} \quad (6.2.1)$$

for small enough δ . The first expression on the right-hand side is bounded by

$$\begin{aligned} & \left| \int_{M_F^r(\delta, T_0)} \kappa^r d\mu^r - \int_{M_r(\delta, T_0)} \kappa d\mu \right| \\ & + \int_{M_C^r \cap (B_\delta(T_0))^C} \kappa^r d\mu^r + \int_{M(\delta, T_0) \setminus M_r(\delta, T_0)} \kappa d\mu \\ & \leq \left| \int_{M_F^r(\delta, T_0)} \kappa^r d\mu^r - \int_{M_r(\delta, T_0)} \kappa d\mu \right| + \frac{\varepsilon}{6}. \end{aligned} \quad (6.2.2)$$

Since $M_C^r \cap (B_\delta(T_0))^C \subset M_C^r(\delta)$ Lemma 6.7 implies that the next to the last integral on the left hand side is less than $\varepsilon/12$ for small enough r . Since $M(\delta, T_0) \setminus M_2(\delta, r, T_0) \subset M \setminus M_r$, by Lemma 3.4 the last integral is bounded by the same number.

Let $M_G(\delta, r, T_0) = M_r(\delta, T_0) \cap \pi^r(M_F^r(\delta, T_0))$. Then

$$\begin{aligned} & \left| \int_{M_F^r(\delta, T_0)} \kappa^r d\mu^r - \int_{M_r(\delta, T_0)} \kappa d\mu \right| \\ & \leq \left| \int_{\pi^{-r}(M_G(\delta, r, T_0))} \kappa^r d\mu^r - \int_{M_G(\delta, r, T_0)} \kappa d\mu \right| \\ & \quad + \int_{M_F^r(\delta, T_0) \setminus \pi^{-r}(M_G(\delta, r, T_0))} \kappa^2 d\mu^r + \int_{M_r(\delta, T_0) \setminus M_G(\delta, r, T_0)} \kappa d\mu. \end{aligned} \quad (6.2.3)$$

Here π^{-r} is the inverse of π^r . Clearly $\mu^r(M_F^r(\delta, T_0) \setminus \pi^{-r}(M_G(\delta, r, T_0))) \rightarrow 0$ as $r \downarrow 0$ and since $\kappa^r(x)$ is uniformly bounded for $x \in M_F^r(\delta, T_0)$ (by Lemmas 6.8 and 6.9) the next to the last integral is $< \varepsilon/12$. Analogously $\mu(M_r(\delta, T_0) \setminus M_G(\delta, r, T_0)) \rightarrow 0$ and since κ is uniformly bounded on $M_r(\delta, T_0) \forall r > 0$ the last expression is less than $\varepsilon/12$ as well.

Next notice that

$$\begin{aligned} & \left| \int_{\pi^{-r}(M_G(\delta, r, T_0))} \kappa^r d\mu^r - \int_{M_G(\delta, r, T_0)} \kappa d\mu \right| \\ & \leq \int_{M_G(\delta, r, T_0)} |\kappa^r(\pi^{-r}(x)) - \kappa(x)| \mu(dx) \\ & \quad + \sup_{x \in M_G(\delta, r, T_0)} \kappa^r(x) \|\mu - \mu^r \pi^{-r}\| \end{aligned} \quad (6.2.4)$$

where the total variation norm is on $M_G(\delta, r, T_0)$. But

$$\|\mu - \mu^r \pi^{-r}\| \leq \|\mu - \mu^r \pi^{-r}\|_{TV, M_r} \rightarrow 0$$

and the last term in (6.2.4) is less than $\varepsilon/6$.

Lemmas 3.3 and 6.4 will then imply that $|\kappa(x) - \kappa_{2n}(x)| < 1/T_c \leq \varepsilon/18$ and $|\kappa^r(\pi^{-r}(x)) - \kappa_{2n}^r(\pi^{-r}(x))| < \varepsilon/18 \forall x \in M_G(\delta, r, T_0)$ for all small r . Hence

$$\int_{M_G(\delta, r, T_0)} |\kappa^r - \kappa| d\mu \leq \int_{M_G(\delta, r, T_0)} |\kappa_{2n}^r - \kappa_{2n}| d\mu + \frac{\varepsilon}{9}. \quad (6.2.5)$$

By Lemma 6.10 the last integral is bounded by $\varepsilon/18$ for small r . Formulas 6.2.1–5 imply $|\lambda'_+ - \lambda_+| < \varepsilon$. $\square$

7. α -congruence in the smooth case and comparisons

We now extend the theorem in § 6 for smooth billiards into an α -congruence result with an argument very similar to that in § 4. We will not reprove the result but rather just point out the differences. Now the α -congruence will compare with structural stability in an interesting way which we discuss subsequently.

THEOREM 7.1. *The dynamical systems (M, S_t, μ) and $(M, \pi^r \circ G_{ct}^r \circ \pi^{-r}, \mu^r \circ \pi^{-r})$ are α -congruent for small r and some c close to 1.*

Recall that π^r is the projection from M^r onto M and $\pi^{-r} = (\pi^r)^{-1}$. The result has the following interpretation: The standard and the smooth billiard are indistinguishable flows for an α -reliable viewer that has the resolution α . Note that we assume that the viewer cannot distinguish between the top and bottom sheets of the manifold M^r which seems natural since their distance approaches zero. If the manifold is 'opaque' the theorem must be altered to indicate the α -congruence of the standard billiard and the Benoulli factor of the geodesic flow obtained by factoring a two-point suspension away.

The proof of Theorem 4.2 applies to Theorem 7.1 with obvious changes. In fact it is simplified since the phase spaces are identical. Lemma 4.3 holds verbatim in this case. We note that the expanding fibres of the smooth billiard do not have singularities. But since they fold from one sheet to the other whenever a ray bundle grazes the regions M_C^r , π^r -projections of the fibres can double as in SDB. Therefore the argument of Lemma 4.3 applies to π^r -projections of the fibres of smooth billiards as well (their length estimates are identical). Given any t_0 then for some $r_0 > 0$ we get $|h(S_{t_0}, \mathcal{P}) - h(G_{t_0}^r, \mathcal{P}^r)| < \varepsilon \forall r \leq r_0$ where $\mathcal{P}^r = \pi^{-r}(\mathcal{P})$ is a partition on M^r and $\mathcal{P}$ is a regular partition on M .

7.2. The smooth dispersive billiard is not a structurally stable perturbation of the standard dispersive billiard. This follows from the simple observation that the inside parallels of the necks in the SSDB cannot be mapped to trajectories of the SDB with a homeomorphism that is close to identity. In fact the structural stability fails in a very strong way since it is true only for a set of trajectories of measure zero. This is because for any generic geodesic trajectory $\{G_t x\}_{t=-\infty}^{\infty}$ and any finite $T > 0$ we can find an interval of length T such that $G_t x$ stays continuously for this duration of time in M_C . But then if T is large enough the trajectory cannot be uniformly close to a SDB trajectory. Since α -congruence allows this kind of excursions of coupled trajectories for a set of times of positive density Theorem 7.1 is possible.

Our result indicates that once certain rigid and purely geometric requirements are relaxed a dispersive billiard and a geodesic flow on a manifold of equal non-positive total curvature can in certain cases be considered as almost identical dynamical systems. As long as the entropy computations are possible we expect our type of results to apply to more general manifolds as well. It is also noteworthy that α -congruence does seem to apply to a considerably larger family of dynamical systems than structural stability. Also for the latter the allowable perturbations have to be considered rather carefully (from our example we see that for example not all Hamiltonian perturbations work for the billiard although this is a Hamiltonian system). And in some cases like the boundary perturbation of billiards even the correct extension of the definition of the structural stability remains still open.

Acknowledgements. This work is part of a PhD thesis written at Stanford University. The author wishes to thank professor Donald Ornstein for numerous helpful conversations on these topics and the referee for pointing out inaccuracies in the original manuscript.

Appendix

Definition. Let $\mathcal{P}$ and $\bar{\mathcal{P}}$ be a finite partitions of same cardinality on a space X . Suppose $(f, \mathcal{P})$ and $(\bar{f}, \bar{\mathcal{P}})$ be two processes on X and let

$$d_T((f, \mathcal{P}), (\bar{f}, \bar{\mathcal{P}}))(x, y) = \frac{1}{T} \int_0^T d(\mathcal{P}(f_t x), \bar{\mathcal{P}}(\bar{f}_t y)) dt$$

where d is some metric on the indices of the atoms. If the processes are ergodic, the $\bar{d}$ -distance can be defined as

$$\bar{d}((f, \mathcal{P}), (\bar{f}, \bar{\mathcal{P}})) = \liminf_{T \rightarrow \infty} d_T((f, \mathcal{P}), (\bar{f}, \bar{\mathcal{P}}))(x, \iota(x))$$

where ι where ι is an isomorphism. The limit can be shown to be a.s. independent of x . The definition readily extends to the case where $\mathcal{P} = \bar{\mathcal{P}} = \bigcup_{x \in X} \{x\}$ and d is the metric on X . Finally the discrete case is defined by replacing integration by summation.

Definition. The atoms of the partition $\bigvee_{i=0}^{N-1} T^{-i}\mathcal{P}$ which are of the form $\bigcap_{i=0}^{N-1} T^{-i}P_{j_i}$ are called *future N -P-atoms* and the sequence $\{j_i\}_{i=0}^{N-1}$ is the corresponding *N -P-name*.

Definition. Let $\mathcal{P}$ and $\bar{\mathcal{P}}$ be as above and let $\text{dist}(\mathcal{P}, \bar{\mathcal{P}}) = \sum |m(P_i) - m(\bar{P}_i)|$ where the m is an underlying measure on X or for example the invariant measure of a process. A process $(T, \mathcal{P})$ on the space X is *finitely determined* if $\forall \varepsilon > 0 \exists N, \delta > 0$ such that

$$\text{dist}\left(\bigvee_{i=0}^{N-1} T^{-i}\mathcal{P}, \bigvee_{i=0}^{N-1} \bar{T}^{-i}\bar{\mathcal{P}}\right) < \delta \quad \text{and} \quad |h(T, \mathcal{P}) - h(\bar{T}, \bar{\mathcal{P}})| < \delta$$

imply

$$\bar{d}((T, \mathcal{P}), (\bar{T}, \bar{\mathcal{P}})) < \varepsilon.$$

REFERENCES

- [AA] V. I. Arnold & A. Avez. *Ergodic Problems of Classical Mechanics* W. A. Benjamin (1968).
- [BS] L. A. Bunimovich & Ya. G. Sinai. On a fundamental theorem in the theory of dispersive billiards. *Math. USSR Sb.* **19** (1973), 407-424.
- [BS†] G. Benettin & J.-M. Strelcyn. Numerical experiments on the free motion of a point mass moving in a plane convex region: Stochastic transition and entropy. *Phys. Rev. A* **17** (1978), 773-785.
- [Eb] P. Eberlein. When is a geodesic flow of Anosov type I? *J. Diff. Geom.* **8** (1973), 437-463.
- [El] K. Eloranta. Alpha-congruence for Markov processes. *Ann. Prob.* **18**(4) (1990), 1583-1601.
- [GO] G. Gallavotti & D. S. Ornstein. The billiard flow with a convex scatterer is Bernoulli. *Comm. Math. Phys.* **38** (1974), 83-101.
- [Ke] G. Keller. *Ergodizität und K-eigenschaft der Kollisions Transformation beim Sinai-Billiard mit Endlichem Horizont*, Thesis, Mathematische Institut, University of Erlangen (1977).
- [Kh] A. N. Khovanskii. *The Application of Continued Fractions*. P. Noordhoff (1963).
- [KS] A. Katok & J.-M. Strelcyn. *Invariant manifolds, entropy and billiards: Smooth maps with singularities. Lecture Notes in Mathematics* **1222**, Springer (1986).
- [M] A. Manning. Curvature bounds for the entropy of the geodesic flow on a surface. *J. London Math. Soc.* (2), **24** (1981), 351-7.
- [O] D. S. Ornstein. *Ergodic Theory, Randomness and Dynamical Systems*. Yale (1974).
- [OW] D. S. Ornstein & B. Weiss. Statistical properties of chaotic systems. *Bull. Amer. Math. Soc.* **24** (1) (1991), 11-16.

- [P] Ya. B. Pesin. Characteristic Lyapunov exponents and smooth ergodic theory. *Russian Math. Surveys* **32**(4) (1977), 55–114.
- [S1] Ya. G. Sinai. Dynamical systems with elastic reflections. *Russian Math. Surveys* **25**(2) (1970), 137–189.
- [S2] Ya. G. Sinai. Classical dynamical systems with countable Lebesgue spectrum II. *Amer. Math. Soc. Transl.* (2) **68** (1968), 34–88.
- [Sp] M. Spivak. *A Comprehensive Introduction to Differential Geometry*, Vol. 3, Publish or Perish (1979).
- [W] M. Wojtkowski. Principles for the design of billiards with nonvanishing Lyapunov exponents. *Comm. Math. Phys.* **105** (1986), 391–414.

