DISPERSIVE BILLIARDS AN BROWNIAN MOTION ARE STATISTICALLY INDISTINGUISHABLE

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Abstract: *We relax the uniform Doeblin condition in the $\bar{d}$-convergence theorem in [E1] into milder and more explicit conditions. In doing this we also present a general $\bar{d}$-convergence theorem for processes with jump distributions being arbitrary mixtures of absolutely contnuous and discrete type and treat the cases of compact and noncompact statespace in a unified way. The applicability of the theorem is thus further enchanced which we illustrate by extending a result by Bunimovich and Sinai on billiards. In particular the strengthening of their invariance principle paves the road to general stabilty results in the sense of the α-congruence.*

Key words and phrases: Invariance principle, $\bar{d}$-metric, α-congruence, Bernoulli process, billards

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Introduction

To analyze the stability properties of various stochastic as well as deterministic chaotic systems a novel approach was presented by Ornstein and Weiss in [**OW**] and [**O2**]. A new type of infinite-time stability called α-congruence was defined and shown to have most of the characteristics of structural stability. Moreover it accomodates a wider class of systems than just smooth flows and has a number of features of physical importance that the latter stability concept lacks. Intuitively one might describe the notion as follows: two dynamical systems are α-congruent if they are measure theoretically isomorphic and the isomorphism moves all but α of the statespace (in the sense of measure) by less than α.

Stability in the sense of α-congruence has been shown to a number of systems (see [**OW**], [**E1**] and [**E2**]). When the limiting Bernoulli process is of infinite entropy the key ingredient in the stability analysis is the calculation of the $\bar{d}$-distance between the processes. In discrete time this metric is the infinite-time generalization of the Hamming distance. In this paper we generalize this argument by removing the uniform Doeblin condition of [**E1**] and intoducing the uniformity in a slight strengthening of the invariance principle (its existence is a necessary condition). The approximating process is allowed to have a general jump distribution and the requirement for compactness of the space is removed.

As an illustration of the consequencies of the result for other systems than the random walks considered earlier we extend an invariance principle by Bunimovich and Sinai ([**BS**]). We show that a billiard ball moving on a periodic two dimensional table with round obstacles once scaled affinely fine enough yields a motion $\bar{d}$-close to a Brownian motion. This in turn can easily be converted into an α-congruence statement about the statistical indistinguishability of these two systems.

1. The $\bar{d}$-convergence

Let (M, d) be a Polish space. By (X_t, P) we denote a non-degenerate diffusion process on (M, d) that has the stationary distribution $\lambda \in \mathcal{P}(M)$. Hence by tightness the process "essentially" lives on a compact set. The diffusion satisfies the property that for any compact $C \subset M$ we have $\sup_{x \in C} \|P_x(t) - \lambda\|_{TV} \to 0$ as $t \to \infty$.

If M is compact the convergence above can be verified by checking e.g. the Doeblin condition. The rate at which zero is approached turns out then to be exponential the exponent being the largest non-zero eigenvalue of the generator of X (see e.g. [**F**]).

Let $\{(X_t^n, P^n)\}_{n \geq 1}$ be a sequence of stationary stochastic processes on (M, d) with stationary distributions $\{\lambda^n\}$ in $\mathcal{P}(M)$. We assume that for P^n-a.e. ω the paths of the process are right continuous and have left limits as $t \in \mathbb{R}_+$ i.e. belong to D. The function space equipped with the Skorokhod topology is separable. In our notation $P_x^n(T)$ denotes the one-dimensional marginal and $P_x^n([0,T))$ the measure on paths on $D([0,T), M)$ that started at $X_0 = x$. If the initial values are unspecified they are assumed to be drawn from the stationary distributions.

We assume the following kind of uniformity in the weak convergence of the P^n-sequence. Let A be a λ-continuity set and $T > 0$. Then $\forall \epsilon > 0\ \exists n_1(T, A)$ such that $\forall n \geq n_1\ |P_x^n(T)(A) - P_x(T)(A)| < \epsilon$ except for $x \in F^n$ such that $\lambda^n(F^n) < \epsilon$. We call this *condition (E)* due to its resemblance to Egorov's Theorem. It is easy to see that if $\lambda^n \ll \lambda$ and $[d\lambda^n/d\lambda] \leq M$ hold λ-a.s. for some finite M, then (E) holds. We formulate (E) to accomodate a general jump distribution for P^n.

Let

$$d_T(X^n, X) = \frac{1}{T}\int_0^T d(X_t^n, X_t)dt$$

and define the following analog of the Prohorov metric:

$$\bar{d}_T(P_{x_1}^n, P_{x_2}) = \inf_{\tilde{\mu}} \inf\{\epsilon > 0 |\ \tilde{\mu}(\{\omega|\ d_T(X^n, X) \geq \epsilon\}) \leq \epsilon\}.$$

Here $\tilde{\mu}$ is a coupling measure with marginals $P_{x_1}^n([0,T))$ and $P_{x_2}([0,T))$. Finally let

$$\bar{d}(P_{x_1}^n, P_{x_2}) = \sup_{T \geq 0} \bar{d}_T(P_{x_1}^n, P_{x_2}) = \lim_{T \to \infty} \bar{d}_T(P_{x_1}^n, P_{x_2}).$$

The last equality is shown e.g. in **[O1]**.

The main result can now be stated.

Theorem: *Suppose* $\{(X_t^n, P^n)\}_{n \geq 1}$ *and* (X_t, P) *are defined as above and that* $P_{x^n}^n \Rightarrow P_x$ *as* $x^n \to x$. *Then* $\bar{d}(P^n, P) \to 0$.

Remark: The weak convergence is necessary but not sufficient condition (as shown in **[E1]**).

Proof: Step 0: Choose a compact set C such that $\lambda(C) > 1 - \epsilon$. If M is compact let $C = M$.

Step 1: By using the convergence property of the diffusion we obtain T_u such that

$$\sup_{x \in C} \|P_x(T_u) - \lambda\| < \epsilon.$$

Hence

$$\sup_{x,y \in C} \|P_x(T_u) - P_y(T_u)\| < 2\epsilon.$$

Then choose a bounded set C', $C \subset C' \subset M$ such that

$$\inf_{x \in C} P_x(X_t \in C' \ \forall t \in [0, T_u)) > 1 - \epsilon.$$

Let $D = diam(C')$. Choose a coupling time T_c so that $DT_u/(T_u + T_c) < \epsilon$.

Step 2: By the invariance principle $P^n_{x^n}([0, T_c)) \Rightarrow P_x([0, T_c))$ if $n \to \infty$ and $x^n \to x$. Let ρ_{T_c} be the Prohorov metric on the space of measures on $D([0, T_c), M)$. The function space is separable hence $\rho_{T_c}(P^n_{x^n}, P_x) \to 0$. By using Egorov's Theorem twice we get that for all $\delta \le \delta_0$ and $n \ge n_0$ it holds that $\rho_{T_c}(P^n_{x^n}, P_x) < \epsilon$ if $d(x^n, x) < \delta_0$ and $x \notin F$, $\lambda(F) < \epsilon$. Let ${}_{\mathbf{x}}\eta^n_{[0,T_c)}$ denote the corresponding coupling measure. If $d(x^n, x) \ge \delta_0$ or $\mathbf{x} = (x^n, x) \in M \times F$ let the coupling be independent.

Step 3: Let $\mathcal{P}$ be a N atom partition of $C \setminus F$ such that P_i's are λ-continuity sets of positive measure and diameter less than $\delta_0/2$. Let $P_0 = (C \setminus F)^c$. Also define the following pseudonorm (total variation on $\mathcal{P}$):

$$\|m_1 - m_2\|_{\mathcal{P}} = \sum_{i=1}^{N} |m_1(P_i) - m_2(P_i)|.$$

Clearly $\|m_1 - m_2\|_{\mathcal{P}} \le \|m_1 - m_2\|_{\mathcal{P}'} \le \|m_1 - m_2\|$ if $\mathcal{P} \subset \mathcal{P}' \subset \mathcal{B}$.

Step 4: From the condition (E) we get the existence of $n_1(T_u, \mathcal{P})$ such that $\|P^n_{x^n}(T_u) - P_{x^n}(T_u)\|_{\mathcal{P}} < 2\epsilon$ as $n \ge n_1$ for all $x^n \notin F^n$, $\lambda^n(F^n) < \epsilon$. Combination of this with Step 1 and the ordering of the norms as above yields

$$\sup_{\substack{x^n \in C \setminus F^n \\ x \in C}} \|P^n_{x^n}(T_u) - P_x(T_u)\|_{\mathcal{P}} < 4\epsilon.$$

Define for $x_1 \in C \setminus F^n$, $x_2 \in C$

$$\begin{aligned} &\rho_{\mathcal{P}}\left(P^n_{x_1}([0, T_u)), P_{x_2}([0, T_u))\right) \\ &\quad = \inf_{\nu} \inf\{\Delta > 0 |\ \nu(\{\omega |\ X^n_{T_u} \text{ and } X_{T_u} \text{ are in different} \\ &\qquad\qquad P_i \text{ or one or both are in } P_0\}) < \Delta\} \end{aligned}$$

and call the optimal coupling ${}_{\mathbf{x}}\nu^n_{T_u}$. Since the total variation bounds twice the coupling error we have the supremum of $\rho_{\mathcal{P}}$ over the given set to be less than 2ϵ. Off $\{C \setminus F^n\} \times C$ the coupling is independent.

Step 5: Define a (Markovian) coupling on paths on $[0, T)$, $T = T_u + T_c$ from the constructed measures by

$${}_{\mathbf{x}}\mu^n_{[0,T)} = \int {}_{\mathbf{x}}\nu^n_{T_u}(\mathbf{dz})_{\mathbf{z}}\nu^n_{[0,T_c)}.$$

We next show that this is a good coupling on $[0,T)$ in the sense of average distance between the paths. Now

$$d_T(X^n, X) \leq \frac{T_u}{T} d_{T_u}(X^n, X) + \frac{1}{T_c}\int_{T_u}^{T_u+T_c} d(X_t^n, X_t)dt$$

and we denote the first and second terms of the right hand side by I and II respectively. Clearly

$$I \leq \frac{T_u}{T}\{d_{T_u}(X^n, C) + diam(C) + d_{T_u}(X, C)\}.$$

From the rarity of the excursions we get that

$$_{\mathbf{x}}\mu^n_{[0,T)}\left(\frac{T_u}{T} d_{T_u}(X, C) > \frac{T_u}{T} D\right) = P_x(d_{T_u}(X, C) > D) < \epsilon \qquad \forall x \in C$$

and by the invariance principle a similar estimate holds for the perturbation as well. Step 1 bounds the rest of I.

In the case of bounded M the bound for II is immediate. In the general case we first write for $\mathbf{x} \in \{C \setminus F^n\} \times C$

$$_{\mathbf{x}}\mu^n_{[0,T)} = \int_{z\in P_0} + \int_{z\in \cup_1^N P_i} {}_{\mathbf{x}}\nu^n_{T_u}(dz^n, dz)_{\mathbf{z}}\eta^n_{[0,T_c)}.$$

On the set $E(\epsilon) = \{\,\omega \mid \sup_{t\in[T_u,T)} d(X^n, X) > \epsilon \mid (X_0^n, X_0) = \mathbf{x}\}$ the first integral is bounded by $c\epsilon$ since the second marginal of $\nu^n_{T_u}$ is absolutely continuous with respect to λ and the set P_0 is small by steps 0 and 2. The constant c can subsequently change from line to line but it is independent of n. By the fact that the $\mathcal{P}$-variation is small for the chosen $\mathbf{x}$ we get that over $E(\epsilon)$

$$\begin{aligned}
&\int_{z\in \cup_1^N P_i} {}_{\mathbf{x}}\nu^n_{T_u}(\mathbf{dz})_{\mathbf{z}}\eta^n_{[0,T_c)} \\
&\quad = \sum_{i=1}^{N}\left\{\int_{z^n, z\in P_i} + \int_{z^n\notin P_i, z\in P_i} {}_{\mathbf{x}}\nu^n_{T_u}(\mathbf{dz})_{\mathbf{z}}\eta^n_{[0,T_c)}\right\} \\
&\quad \leq \sum_{i=1}^{N}\left\{c\epsilon\int_{z^n, z\in P_i} {}_{\mathbf{x}}\nu^n_{T_u}(\mathbf{dz}) + c\epsilon\int_{z^n\notin P_i, z\in P_i} {}_{\mathbf{z}}\eta^n_{[0,T_c)}\right\} \leq c\epsilon.
\end{aligned}$$

Since the average distance can not exceed the supremum of the distance we get that $\bar{d}_T(P^n_{x_1}, P_{x_2}) < c\epsilon$. Furthermore by the choice of C and F^n we have $\bar{d}_T(P^n_{\lambda^n}, P_\lambda) < c\epsilon$. Call the corresponding $c\epsilon$-good coupling μ^n_T.

Step 6: By induction we get a family of couplings

$$\mu^n_{kT} = \int \mu^n_{(k-1)T}(\mathbf{dx})_{\mathbf{x}}\mu^n_{[0,T)}.$$

Clearly these are again Markovian. Furthermore denote the limit coupling by μ^n_∞.

Let us now consider the dynamical system $(D \times D, \mathcal{D}, \theta^n_T \times \theta_T, \mu^n_\infty)$. As usual θ_t's are the shifts along paths. By a standard argument the product can be chosen to be ergodic (e.g. **[O1]**). But then by the Ergodic Theorem

$$\frac{1}{kT}\int_0^{kT} d(X^n_t, X_t)dt = \frac{1}{k}\sum_{i=0}^{k-1} d_T((X^n, X)\circ(\theta^n_{iT}, \theta_{iT}))$$
$$\longrightarrow \int_{D\times D} d_T(\mathbf{x})\mu^n_T(\mathbf{dx}) \qquad \mu^n_\infty\text{—a.s..}$$

By Step 5 the last expression is bounded by $c\epsilon$. Therefore the proof is complete. ∎

2. Application and extension

Our result is directly applicable to all the cases considered in **[E1]**. But since the requirement for the uniformity in the tail of the random walk sequence is now removed we can in fact expect a wider range of applications. This includes sequences of dependent random walks for which an invariance principle is known (see e.g. **[EK]**) as well as deterministic dynamical systems with only a minimal amount of "seed" randomness in them. To illustrate this more explicitly we consider the case of billiards. Our system is slightly modified from that by Bunimovich and Sinai (**[BS]**).

Let us consider the uniform motion (constant speed) of a point particle in the plane with a configuration of scatterers. The collisions to the scatterers are elastic and obey the principle of "angle of incidence equals the angle of reflection". The scatterers are disks of arbitrary diameter and the configuration is invariant under a discrete subgroup of translations of the plane (Γ) with compact fundamental domain. In particular we can choose the fundamental domain to be a rectangle $\Pi = \{\mathbf{q} | \ 0 \le q_1 < B_1, \ 0 \le q_2 < B_2\}$. The arrangement of the scatterers is also assumed to have finite horizon i.e. the length of the longest straight line that avoids the scatterers is bounded. In the phase space $M = \{\mathbf{x} = (\mathbf{q}, \mathbf{p}) \in \Gamma\Pi \backslash \{scatterers\} \times S^1\}$ the billiard flow is defined as $\mathbf{x}(t) = (\mathbf{q}(t), \mathbf{p}(t)) = S_t\mathbf{x}$ where $\mathbf{q}$ is continuous and $\mathbf{p}$ the right continuous version with left limits. S preserves the Lebesque measure. Let the initial value be chosen according to a probability measure which is absolutely continuous with respect to the Lebesque measure and supported in $M \cap \{\Pi \times S^1\}$.

Define now a sequence of scaled planar motions as

$$q^n(t) = \frac{1}{n} q(n^2 t) \qquad (mod\ B_1, B_2), \quad t \geq 0$$

i.e. we are observing the position of the billiard ball through the toral window Π. If then μ^n denotes the measure on the paths for $t \in [0, T]$ and μ is the law of the Brownian motion on Π Bunimovich and Sinai have the the beautiful result:

Theorem: $\mu^n \Rightarrow \mu$.

Since $\Delta\phi = 0$ on the torus Π together with $\int \phi dm = 1$ imply that ϕ =const> 0 it is easy to see that the Brownian Motion satisfies the Doeblin condition and in fact an exponential convergence is attained. Moreover λ^n is Lebesque measure and its Radon-Nikodym derivative with respect to λ is clearly bounded. Hence

Corollary: $\bar{d}(\mu^n, \mu) \to 0$.

In fact even this result can be slightly improved. By **[E2]** the type of billiard considered above is stable under smooth perturbations of the obstacles. Hence we can remove the rigid assumption of circular scatterers and replace it by convex C^3 obstacles that have their boundary curves C^2-approaching circles. If a system with this type of table is scaled as above the $\bar{d}$-convergence result holds again. It appers that the convexity and $C^{2+\alpha}$-smoothness are the relevant features of the scatterers and the rest of the assumptions are just for computational simplicity.

The $\bar{d}$-convergence argument above can under general Bernoulliness conditions be extended to α-congruence. We restrict ourselves here just to illustrate this extension in the particular application at hand. Already there the strength of the concept as well as its intuitive appeal will become evident. For further elaboration see **[OW]**.

Definition: *Two measure preserving flows (M, f_t, μ) and $(M, \bar{f}_t, \bar{\mu})$ on a compact metric space (M, d) are α-congruent if there is an isomorphism ι (i.e. an invertible measure-preserving transformation such that $\iota \circ f_t = \bar{f}_t \circ \iota$) between them and ι moves all but α of the points of M by less than α i.e. $\mu(\{x \in M | \ d(\iota(x), x) \geq \alpha\}) < \alpha$.*

If the flows are also ergodic then by the Ergodic Theorem

$$\frac{1}{T} \int_0^T d((\bar{f}_t \circ \iota)(x), f_t(x)) dt \longrightarrow \int_M d(\iota(x), x) d\mu \qquad \mu\text{—a.e.}$$

i.e. the definition is equivalent to requiring that (in addition to the isomorphism) the infinite trajectories of f_t and $\bar{f}_t$ are almost surely within α of each other for all except density α of times.

Let us form a direct product of the billiard flow with an infinite-entropy Bernoulli system. We can interpret this as an observation of the billiard by a randomly perturbed viewer. Suppose that these perturbations are confined into a disk of size ϵ on (M, d) i.e. the observer has a resolution bound/observation error of size ϵ. This system is isomorphic with the Brownian flow since they are Bernoulli systems with equal entropy. Now let n be so large that the billiard and the Brownian flow are ϵ-close in $\overline{d}$. Then the viewed billiard and the Brownian flow are 2ϵ-close in $\overline{d}$-sense by the independence of the perturbation sequence. Hence they are 2ϵ-congruent. *In other words an observer that commits occasional observation errors and has finite resolution is forever unable to distinguish a fine enough billiard from the Brownian motion.* We find this striking especially when understood that the billiard table can be arbitrarily densely packed with scatterers in which case the billiard lives on a tiny fraction of the statespace of the Brownian particle.

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