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Cellular automata for contour dynamics

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Abstract

The dynamics of deterministic partially permutive cellular automata in two dimensions is considered. In particular we lay out the design principles and analyze the dynamics of generalized voter-type rules. They provide a two parameter model class of cellular automata which exhibit annealing, diffusive and critical domain behavior. We analyze the dynamics rigorously in certain cases and also show that all types of behavior are very close to their ideal counterparts observed in probabilistic models based on strong independence assumptions. This is further evidence that the statistical mechanics extends to cover also classes of purely deterministic dynamics.

Keywords: Cellular automaton; Partial permutivity; Voter-model; Annealing and critical dynamics

0. Introduction

For already some time it has been known that deterministic cellular automata (c.a.) are capable of exhibiting dynamics intriguingly close to that observed in statistical mechanical systems. Indeed stimulated by this there has been a growing interest to engineer c.a. with properties imitating diffusion, phase transitions, fluid flow, crystal growth, etc. For general reference see [12].

Since c.a. are by their nature quite easy to implement and extremely efficient computationally (especially in parallel environment) it is not surprising that most c.a. studies so far have heavily relied on simulations. Through these a number of useful rules have emerged. But their theoretical foundations are by and large missing except for a few cases mostly in the lattice gas context (e.g. [7]).

In this note we provide some theoretical insight to the construction of deterministic two-dimensional c.a. with exactly the desired domain interaction/contour dynamics. These are c.a. in the original sense i.e. discrete update schemes on a lattice and they do not involve any particles like in lattice gas automata. The unifying idea of partial permutivity has been elaborated in one-dimensional c.a. elsewhere (see [3]) and here we extend it to two dimensions. The interest is of course to see the deeper phenomena one may encounter in the

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case of a non-trivial neighborhood. Motivated by statistical mechanics we restrict ourselves to rules exhibiting (pseudo) random contour dynamics [1]. Some of the rules are close relatives of the probabilistic voter-models in interacting particle systems recently studied by many (e.g. [2,10] and references therein). C.a. models for these have been considered earlier, e.g. in [14], but our approach is quite different. It connects the problem to advances in general lattice actions and provides some rigor to the approach.

In summary we present a two-parameter family of c.a. rules which exhibit annealing, diffusive and critical behavior depending on where we are in the parameter space. The behavior is identical to that observed in many classical models based on heavy probabilistic assumptions (independent random flipping of spins etc.) which are absent from our basic set-up. The dynamics is continuously dependent on the parameters and we believe that the parametrization is also physically natural. The former property which is novel in the context of c.a. where rules are most often isolated examples encourages us to view our results also as design principles.

The flow of the presentation is as follows. We first introduce the principles of the rules having the necessary stationarity and mixing properties. The first alternatives for the symbolic structure governing the type of the dynamics are laid out in Section 2.1. This is followed by the analysis of the annealing and diffusive dynamics. In Section 2.3 we consider the global dynamical properties as well as aspects of the implementation of the rules. In the conclusion of the chapter point, defects as well as diffusive domain dynamics are discussed. The third chapter introduces deterministic rules with positive temperature. The basic rules are extended to a family of c.a. which exhibit Ising-like critical dynamics and the complete phase diagram is unveiled. As the exposition is aimed at a varied audience the proofs appear in the appendix and in the sections we only record the results.

1. The basic set-up

Let $S = \{0, 1, \ldots, |S| - 1\}$, $|S| < \infty$, be a set of **symbols**, the **alphabet**. Let L be the square lattice $\mathbf{Z}^2$ and $L^{(1/2)}$ its dual lattice $(\mathbf{Z} + 1/2)^2$. The sets $X = S^L$ and $X^{(1/2)} = S^{L^{(1/2)}}$ are the sets of **configurations**. On both of these we have the natural coordinate actions, the **horizontal** and **vertical shifts** defined by $(\sigma_h x)_{(j_h, j_v)} = x_{(j_h+1, j_v)}$ and $(\sigma_v x)_{(j_h, j_v)} = x_{(j_h, j_v+1)}$ for $x \in X$.

Definition 1.1. *A* ***block map*** *or a* ***cellular automaton rule*** *is a map on four symbols in a* 2×2 *square* ***neighborhood*** $f : S^4 \to S$. *A* ***cellular automaton*** *is the map* $F : X \to X^{(1/2)}$ *and* $F : X^{(1/2)} \to X$ *obtained by requiring that the cellular automaton rule commutes with the shifts* σ_h *and* σ_v.

Any two-dimensional cellular automaton defined on square or triangular lattice can be transformed to this canonical form (the procedure in [3] extends easily to any dimension). Fig. 1 shows an illustration of the lattices and the cell neighborhood. Occasionally we also refer to a set-up where a pair of c.a. rules alternates between two triangular lattices. The rules are the same upto orientation – the neighborhood for the other is the one shown on the right rotated 180 degrees (for details see [6]).

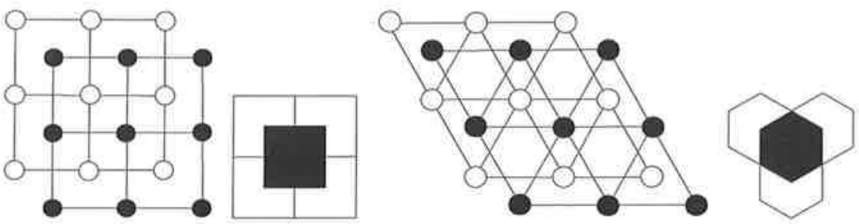

Fig. 1. The alternating square and triangular lattices. Cells in the neighborhoods are white and in updates black.

From this on we use j_h and j_v as spatial indices and i as a temporal index indicating the iteration of the c.a.

The representation given in Definition 1.1 is particularly useful because it enables one to distinguish invariant sets of configurations.

Definition 1.2. *The c.a. map is* **partially permutive** *if it has an* **invariant subalphabet**: *there exists* $S' \subset S$ *such that* $f(s_1, s_2, s_3, S') = S'$ *for any* $s_i \in S'$ *and the same equation holds for any permutation of the arguments in* f.

Clearly global configurations S'^L and $S'^{L^{(1/2)}}$ generated from an invariant subalphabet S' are invariant under F. We call them **ground states** since they play corresponding physical roles in our models (see beginning of Section 2.1).

To clarify the basic structures we accept the following premise.

Assumption 1.3. *The set of symbols partitions into subalphabets of equal size.*

As shown in [3] the case of intersecting subalphabets can be always extended to satisfy the Assumption. The exclusion of the non-invariant symbols just serves to clarify the essence of the interaction between domains generated from different subalphabets. If the Assumption is not valid very interesting things can happen but we refrain here from investigating the possibly "complex" mixture of the subpermutive and other type of behavior.

For the purpose of defining our rules we split every symbol into two parts. For any symbol s in the alphabet, $s = (a, d)$ where $a \in A$ and $d \in D$. $A = \{0, 1, \ldots, p-1\}$ is the set of subalphabets and $D = \{0, 1, \ldots, q-1\}$ is the set of **digits**. In most of the subsequent analysis we will consider the simplest non-trivial set-up: $p = q = 2$.

To streamline the formalism let $\mathbf{s} = (s_1, s_2, s_3, s_4)$, $\mathbf{a} = (a_1, a_2, a_3, a_4)$ and $\mathbf{d} = (d_1, d_2, d_3, d_4)$ be the symbol, subalphabet and digit vectors in a neighborhood. The entries are counted clockwise from the north-west corner. Our rules are of then of the following form:

$$f(\mathbf{s}) = (A(\mathbf{a}, \mathbf{d}), Q(\mathbf{d})). \tag{1.1}$$

The table $\underline{A} = \{A(\mathbf{a}, \mathbf{d})\}$ is called the **assignment array**. As functions the arrays $\underline{A}$ and Q must map as $\underline{A}: S^4 \to A$ and $Q: D^4 \to D$.

This formulation is motivated by the desire to explain contour gas dynamics observed in deterministic two-dimensional c.a. The **contours** are boundaries between domains generated from different subalphabets. One would like to explain why the boundaries can in deterministic c.a. move/merge/split in a seemingly random fashion like is in dynamic Ising models or in e.g. voter-type probabilistic particle systems ([2]).

A stationary background is provided for the c.a. evolution if the map Q has a special structure. Suppose that Q is a generalized **quasigroup**, i.e. a map $Q: D^4 \to D$ such that the entire digit set is invariant in the sense of Definition 1.2. These maps have the property that the global c.a. map P induced by them (Q is a c.a. rule on digits) is permutive and hence preserves the uniform Bernoulli measure: $\mu_D = \mu_D^{(1/2)} \circ P^{-1}$. Here μ_D and $\mu_D^{(1/2)}$ are these measures supported on D^L and $D^{L^{(1/2)}}$, respectively. The preservation is argued here as in the one-dimensional case. Note that μ_D is a non-trivial Bernoulli measure (not a point mass) when $q \geq 2$. This is the reason why we consider the case $q = 2$.

The basic example of such a quasigroup is the one given by the additive rule

$$Q_S: \mathbf{d} \to \sum_{i=1}^{4} d_i \pmod 2.$$

This is a generalization of the one-dimensional rule introduced by Ledrappier in [9]. Denote the analogous additive rule for the triangular lattice by Q_T. These two rules will underlie all our upcoming c.a. since they

provide maximal degree of mixing in the context of deterministic c.a.

In order to state a mixing result fix a quasigroup Q and consider the subshift of finite type $D^{(3)}$ of all allowed digit evolutions under the induced global map P,

$$D^{(3)} = \left\{ \{d_{(j_h,j_v,i)}\} \mid Q\left(d_{(j_h,j_v+1,i)}, d_{(j_h+1,j_v+1,i)}, d_{(j_h+1,j_v,i)}, d_{(j_h,j_v,i)}\right) = d_{(j_h+1/2,j_v+1/2,i+1)} \right\} .$$

Let σ_t be the time-like shift, i.e. shift in the direction of the iteration of the c.a. map,

$$\left(\sigma_t \tilde{d}\right)_{(j_h,j_v,i)} = \tilde{d}_{(j_h+1/2,j_v+1/2,i+1)} \quad \text{for any } \tilde{d} \in D^{(3)} .$$

Recall that the shift on $D^{(3)}$ is mixing and therefore in particular ergodic if

$$\mu^{(3)}\left(\sigma_h^{j_h}\sigma_v^{j_v}\sigma_t^{i}(A) \cap B\right) \longrightarrow \mu^{(3)}(A)\,\mu^{(3)}(B) ,$$

as $|j_h| + |j_v| + |i| \to \infty$ for any measurable A and B in $D^{(3)}$. Here $\mu^{(3)}$ is the unique shift-invariant measure on the subshift induced by the invariant measure μ_D. For a general construction of the inverse limit $\left(D^{(3)}, \sigma_v\sigma_h\sigma_t, \mu^{(3)}\right)$ see e.g. [11].

Theorem 1.4. *The shift-action $(j_h, j_v, i) \mapsto \sigma_v^{j_h}\sigma_h^{j_v}\sigma_t^{i}$ on the subshift $D^{(3)}$ induced by the additive rule Q_S is mixing.*

Remark. See the Appendix for a note on the proof. It makes sense to ask how "chaotic" the digit evolution can be in a c.a. that admits digit stationarity. The Theorem should be viewed as an answer to this: it can be mixing, i.e. asymptotically independent. Note that it cannot be much more as deterministic c.a. are zero entropy systems. Refinements of the result actually tell how manifold the mixing can be (see [8]).

In the forthcoming analysis our main attention is focused on deterministic c.a. rules. They will have one of the quasigroups Q_S or Q_T in them and only the assignment array will vary. The randomness will be provided only in the form of the initial measures. In view of the results above good initial measures μ are such that their projection on digits is maximally disordered,

$$\pi_2\mu = \mu_D . \tag{1.2}$$

Once this condition is satisfied the dynamics of the c.a. is remarkably similar to two probabilistic models which are useful to keep in mind as references. Given a c.a. it's semi-independent model is the model with the same assignment array but where the digit evolution is independent. Note that in $D^{(3)}$ the digits at any fixed time i are independent but the digits between two iterates are not independent (hence the shift e.g. in the time direction cannot be an independent process). In the **independent model** the digits are removed completely but the neighborhood updates have exactly the same probabilities as in the c.a. started with the initial measure satisfying (1.2). Thereby the neighborhood dependencies are reduced, the model is close to classical statistical mechanics models and better allows rigorous analysis.

2. The voter rules

The analysis of the basic rules introduced in Section 2.1 is in Sections 2.2 and 2.3. The former contains the rigorous results while the latter concentrates to the implementation and global dynamics of the rules. The

reading of Section 2.3 does not assume acquaintance with Section 2.2. Sections 2.4 and 2.5 present extensions beyond the basic annealing set-up using similar rules.

2.1. The assignments

Having defined the underlying digit dynamics which is responsible of the pseudorandomness of the evolution we now complete the definition of the rules. First we lay out the principles for the assignment arrays and then give the canonical representatives of them.

We define all the rules on the two-dimensional square lattice and only point out essential differences in the case of the triangular lattice. The rules will have two invariant subalphabets consisting of two symbols each and they partition the whole alphabet. By our earlier convention the subalphabets are indexed by $A = \{0, 1\}$ and the digits by $D = \{0, 1\}$. Hence we have a total of four symbols. For convenience we call the subalphabet with $a = 0$ the "light" subalphabet and the other the "dark" subalphabet. This is also the way they are rendered in the illustrations.

The basic physical principles we are (first) aiming at are

(i) Subalphabet invariance as in Definition 1.2. The rule should preserve both of the subalphabets and thereby global configurations generated by each of them separately. Such configurations should be thought as the ground states as the c.a. will relax towards them. Measures satisfying (1.2) on the two sets of configurations with constant assignment are then pure phases (as the $+/-$ -phases in Ising model below the critical temperature).

(ii) Majority domination. If a subalphabet has a majority in a 2×2 neighborhood, it should dominate in the determination of the update.

(iii) Assignment symmetry. The rule should commute with flipping of the subalphabet assignment: $f \circ \iota = \iota \circ f$ where $\iota((a, d)) = (1 - a, d)$ (and it is applied to each of the arguments). This guarantees that the two subalphabets are functionally identical and only their labels differ.

(iv) Isotropy (on the lattice). Formally this means that the rule f should be invariant under a cyclic permutation of its arguments, i.e. a rotation of the neighborhood (recall that the entries of the vectors **a** and **d** were picked clockwise (`NW,NE,SE,SW`)).

(v) In the case of even representation of subalphabets in the neighborhood the assignment should be uniformly Bernoulli distributed.

The requirements (iii) and (iv) can be simultaneously satisfied only when there is an uneven representation of the two subalphabets in a neighborhood. So in the case of the triangular lattice there is no contradiction. To see the problem in a $\mathbf{Z}^2$-neighborhood consider two arrangements of assignments, $(0, 0, 1, 1)$ and $(1, 1, 0, 0)$. An isotropic block map would give the same assignment to the update in both cases whereas assignment symmetric map would force a flip in the assignment. In the subsequent analysis we impose assignment symmetry first since this property is fundamental in order to obtain rules without any intrinsic bias in favor of one of the subalphabets. Moreover isotropy can be recaptured in an average form as seen below. Note also that all rules with additive quasigroups are automatically isotropic on digits.

The simplest of our c.a. block maps is the **majority voter rule**. Its quasigroup is Q_S and the assignment array is defined as follows. By even representation we mean that the two subalphabets both have two symbols in the neighborhood.

$$A(\mathbf{a},\mathbf{d}) = \begin{cases} a, & \text{if } a_i = a \text{ for at least three } i\text{'s}, \\ a_1 + \sum_{i=1}^{4} d_i \pmod 2, & \text{if even representation and even time}, \\ 1 - a_1 + \sum_{i=1}^{4} d_i \pmod 2, & \text{if even representation and odd time}. \end{cases}$$

By the first part of the definition the action on the subalphabets is permutive and the majority domination explains the name. In the case of even representation of subalphabets in the neighborhood the checkerboard labeling given is just one of the possibilities. The key property that it has is that flipping of any single digit flips the update assignment as well. The entry a_1 appears in the formula to guarantee assignment symmetry – it could be replaced by any function of $\{a_i\}$ commuting with the flip ι. Note that once (1.2) is satisfied the digits are uniformly Bernoulli distributed at any fixed time. Consequently the subalphabets both have probability $1/2$ of winning the update in the case of even representation, i.e. (v) holds.

If we would have simply $a_1 + \sum_{i=1}^{4} d_i \pmod 2$ in the case of even representation the rule would still produce assignments in the two subalphabets with equal probabilities. But only two different symbols would be possible for any even **a**. The given rule produces all of the four symbols with equal probabilities and consequently will not have bias to any direction. It is isotropic in this weaker sense. Both the given and the simpler rule are assignment symmetric.

The (strict) majority voter rule can be relaxed in different ways without immediately changing the qualitative nature of its dynamics. By a p-**voter rule** we denote the deterministic c.a. rule which is defined as the majority voter rule except that in the case of 3 to 1 majority the minority prevails with probability p. Suppose **a** equals to $(a, a, a, 1-a)$ or its cyclic permutation. For these **a** let

$$A(\mathbf{a},\mathbf{d}) = \begin{cases} a, & \text{if } \sum_{i=1}^{4} d_i \notin C_p, \\ 1-a, & \text{if } \sum_{i=1}^{4} d_i \in C_p. \end{cases}$$

The set $C_p \subset \{0,1,2,3,4\}$ is the set of exceptions to the majority voter rule that the p-voter rule has. So it's choice determines the p value. The name of the rule stems from the fact that as soon as the initial condition (1.2) is satisfied, at any neighborhood where the 3-1-representation of the subalphabets occurs, the c.a. exhibits a "pseudorandom" majority voter rule with the minority subalphabet prevailing with the given probability. The inverse of p can be viewed as the degree of co-operation among the majority.

The majority vote rule obviously corresponds to $C_0 = \emptyset$. The p-value for a given set follows immediately from the Binomial Theorem and the assumption that we have the uniform Bernoulli measure on the digits.

Table 1
p-values available in the $q = 2$ -case

p	Sets of exceptional digit sums, C_p
0	$\emptyset$
1/16	{0}, {4}
1/8	{0,4}
1/4	{1}, {3}
5/16	{0,1}, {0,3}, {1,4}, {3,4}
3/8	{2}, {0,1,4}, {0,3,4}
7/16	{0,2}, {2,4}
1/2	{1,3}, {0,2,4}

The available p-values exceeding $1/2$ follow from the given ones since they are symmetric with respect $1/2$ (but note that for them the principle (ii) is violated). As the set of digits D increases so does the density of possible p-values on the interval $[0,1]$.

2.2. *Basic properties of the rules*

We first study the majority voter rule since it yields to certain "topological" arguments that do not work for the more general rules. After this we analyze the probabilistic models for this and other values of p. The results 2.1–2.3 (and their proofs) concerning the majority voter rule appeared in [5] but we rephrase the statements here since otherwise the presentation would be incomplete. The techniques in the proof of Theorem 2.4 provide a clue to the proof of Theorem 2.2. The proofs of the new results are in the Appendix.

Since we will be concentrating on phase boundary phenomena we must first clarify what is meant by the boundary/contour. This is most natural to do by assigning at each lattice site a closed unit cell which belongs to one of the subalphabets. It is centered at the lattice point and its sides are aligned with the coordinate axis. From this the boundaries between domains of different subalphabets are uniquely defined. We consider two domains that only intersect at an isolated point(s) disjoint.

Call a rectangle with sides parallel to the diagonals of $\mathbf{Z}^2$ ($y = \pm x$) a diamond. The **diamond hull** of a bounded set in the plane is the smallest diamond containing the set. So for example the diamond hull of four cells in a 2×2 arrangement centered at the origin is the set $|y| + |x| \leq 2$. If the set is unbounded then its diamond hull is a plane, a half plane or a wedge defined in the obvious way.

Theorem 2.1. *Given a domain generated from an invariant subalphabet suppose its diamond hull does not intersect the diamond hull of any other such domain. Then the domain will remain inside its diamond hull under the iteration of a majority voter rule.*

Remarks. 1. Note that the result is purely topological as no reference is made to the digit distribution in the initial measure and in particular (1.2) is not assumed here. Indeed the result is true for the corresponding independent and semi-independent models as well since the result only hinges on the structure of the assignment array.

2. If the diamond hulls of two domains intersect then it is possible that the domains could merge. But again if the diamond hull of the union of the domains is isolated it is by the Theorem a confining diamond hull.

The Theorem clearly hints towards annealing dynamics. By the structure of the assignments we expect the boundary contour of a bounded domain to perform a random motion. This should eventually allow an even smaller diamond hull to be fitted around the domain. By the Theorem this is an irreversible event. Extending this intuition to more general domains we expect asymptotically the configurations to consist of arbitrarily large homogeneous domains. In other words the c.a. should show relaxation towards one of the two sets of ground states (consisting of configurations generated from one of the subalphabets alone). We now proceed to analyze the mechanism behind this relaxation.

To see the basic domain shrinkage mechanism let us first suppose that we have the lower half of the lattice $L^{(1/2)}$ below the diagonal $y = x$ generated from the subalphabet $S^{(1)}$ and the rest of the lattice sites from $S^{(0)}$. Denote the $S^{(a)}$-domains by $B^{(a)}$. If we now flip the assignments of the cells under the diagonal in the first quadrant we introduce a **boundary defect** at origin. Call the lines $L_l : y = x - 1$ and $L_r : y = x + 1$ the **left and right hull-lines**. Let L be the line parallel to them and at equal distance from both. So now L is the diagonal $y = x$. The defect is at the unique crossing point of L and $\partial B^{(\cdot)}$. We say that this defect has the **off-set**

$L_r(0) - L_l(0) - 1 = +1$.

If we would reflect the domains with respect to $y = x$ the lines L_r and L_l would be swapped. The defect would still be at the origin but in this arrangement we define its off-set as $L_r(0) - L_l(0) + 1 = -1$.

It is clear that under the iteration of the majority rule the left and right hull-lines remain the same and so does the off-set of the defect. Furthermore it turns out that the boundary defects with off-sets ± 1 have simple binary interaction dynamics. If we have a $+1$-defect and a -1-defect on the diagonal $y = x$ they **annihilate** upon collision and in our set-up a staircase boundary with unit steps is left.

To characterize the interaction properties of boundary defects let us consider a particular ensemble of them. Suppose that we have an **alternating chain** of defects on the diagonal. By this we mean a chain where neighboring defects have a common hull-line and every $+1$-defect is between two -1-defects and vice versa except at the ends (in case there is only a finite number of defects).

Theorem 2.2. *Suppose that we have an initial distribution satisfying (1.2) and the assignment is such that we have an alternating chain on the diagonal $y = x$. Under the iteration of the c.a. the defects perform individually symmetric nearest neighbor random walk. Conditioned on not hitting another defect the location of a defect initially at origin is at the ith iterate (X_i, X_i) where $X_i = \sum_{k=1}^{i} \Delta_k$ and $\Delta_k = \pm 1/2$ with equal probabilities. The increments Δ_k are independent. Moreover the defects are independent upto the time of collision with a neighboring defect which results in an annihilation. The defects annihilate each other maximally: if a finite number of them are originally present at most one is eventually left. If there is a positive initial density of defects on the line the density decays proportional to $1/\sqrt{i}$, where i is the iteration index.*

The given defect formulation extends to any nonzero integral off-set. The motion of these defects can be argued analogously to Theorem 2.2 but new phenomena enter if the off-set is not equal to ± 1 or in the case of just ± 1 off-sets we do not have an alternating chain.

Suppose that we have a single defect with off-set o, say $o \geq 2$ on the diagonal. Then under the iteration of the majority voter c.a. the defect may **branch** into two defects with off-sets o_1 and o_2 such that

$$o_1 + o_2 = o \ (*) , \quad o_1 o_2 \geq 1 . \tag{2.1}$$

The off-sets here are counted as before from left and right hull lines of the boundary segments immediately to the left and right of the defect. The branching is **reversible** and the conservation of the off-set (2.1) $(*)$ also holds for the mergings. It is a immediate property of the majority voter rule that the offspring of a branching defect always has an off-set with the same sign as the parent.

If the defects **merging** have off-sets of opposite sign (2.1) $(*)$ still holds. But now $o_1 o_2 \leq -1$ so their merging is **irreversible** by (2.1). A particular case of this is of course the annihilation considered earlier $(o_1 = -o_2 = 1)$. These principles combined can be used to argue the shrinkage of a bounded domain. From this it is not difficult to conclude the following "smoothening of the boundary" result.

Proposition 2.3. *Suppose that we have a finite chain of defects with finite off-sets $\{o_n\}_{n=1}^{N}$. Denote the off-set sequence at the ith iterate by $\{o_n^{(i)}\}_{n=1}^{N(i)}$. Then $o_i^{(i)}$ and $N(i)$ are uniformly bounded and the total off-set $\sum_{n=1}^{N(i)} o_n^{(i)}$ is constant and the total variation of the off-sets $\sum_{n=1}^{N(i)} |o_n^{(i)}|$ is nonincreasing.*

The irreversibility of the mergings of defects with opposite off-sets is the very reason for the annealing behavior of the majority voter c.a. A curved boundary of a domain can be decomposed into a chain of defects with appropriate off-sets. Their irreversible mergings result in the monotone decreasing of the total variation i.e. straightening of the boundary. In the case of a finite island this should imply the eventual shrinkage of the confining diamond hull. Here we only present this result with an added assumption. We call a finite domain

convex if a line segment connecting any two lattice points in it is completely within the domain.

Theorem 2.4. *Suppose that we have an initial distribution satisfying (1.2) and the assignment is such that we have a finite convex island of one subalphabet surrounded by the other. Then the island vanishes almost surely.*

The main difficulties in proving a global annealing result seem essentially combinatorial. Difficulties bypassed in the proof of the above set in when the domain geometry is not sufficiently controlled. Jumps of the boundary walks become correlated through conservation laws. However these correlations seem extremely weak and they do not seem to influence the annealing behavior in any noticeable fashion (as will be seen in Section 2.4).

By construction the p-voter rules should behave in a rather predictable fashion as p increases. However they do not seem to yield to similar rigorous analysis as the strict voter case above. In particular there is no reasonable definition of a boundary random walk. Also a sharp monotonicity result like Theorem 2.1 fails and it is not clear what its weaker variant should be.

To see what to expect in the $p > 0$ -case we consider the independent model and argue a few results indicating the dominant dynamics. We restrict to the situation where there is a finite light island B (i.e. generated from the subalphabet $S^{(0)}$) surrounded by a dark sea and argue the evolution of the area of the island.

The center of a 2×2 neighborhood where there is an even representation of the two subalphabets in a diagonal arrangement is called a **cross point**. According to our earlier convention both domains are disjoint in such neighborhood. Consider the boundary, ∂B, of a finite connected light domain. We traverse it in such a way that the domain will always be to the left of the direction where we are heading. If a point on the boundary is not a cross point and the boundary turns to the left at it we call it an **outside corner** whereas if it turns right we have an **inside corner**. Passing an inside corner we record -1, an outside corner $+1$ and a cross point 0. Adding these together while traversing the entire boundary of B once gives us the **excess** $e = e(\partial B)$.

Let B_0 be the initial finite light domain and let A_i be its area at the ith iterate under the evolution of a p-voter rule. Clearly A_i must be finite for finite i. In general the domain at the ith iterate, B_i, will have a number of components. Let e_j be the excess of the boundary of the jth component.

Theorem 2.5. *Suppose that we have initially a finite domain B_0 generated from the subalphabet $S^{(0)}$. Then under the independent p-voter rule, $0 \leq p \leq 1$, we have for the expected total area of the domain*

$$\mathbf{E}\left(A_{i+1} \mid A_i > 0\right) = A_i - \frac{(1-4p)}{4} \sum_{\partial B_i} e_j \,. \tag{2.2}$$

The first inference that one draws from (2.2) is that the p value $1/4$ corresponds to a "balanced" voter model: the area process is a martingale. No domain shrinkage mechanism prevails here and the annealing-like phenomenon is just a consequence of the fact that extinctions of finite isolated domains are both irreversible and inevitable since the martingale $\{A_i\}$ reaches any non-negative value including zero with positive probability.

In decrypting the formula (2.2) the following result is of some use. Recall that a domain is simply connected if any loop in it can be continuously shrunk into a point.

Proposition 2.6. *If a finite domain B has no cross points then $\sum e_j$ is a an integral multiple of 4. If it furthermore is simply connected (in particular if it is convex) then $\sum e_j = 4$.*
If we apply the majority voter rule to a configuration with a single finite and convex domain then $\sum e_j \geq 4$ until the extinction of the domain. Such domain vanishes almost surely.

Remark. In the last part of the statement the convexity of the domain is just for clarity. The necessary

condition, a weaker but more technical "non-closability" condition, could be defined as suggested by the proof. The vanishing statement is due to E. Speer.

From the Theorem we see that the **drift** of the process $\{A_i\}$, $\mathbf{E}\left(A_{i+1} - A_i |\ A_i > 0\right)$ is critically dependent on the quantity $\sum e_j$. The last part on the Proposition tells us that for the majority voter rule the drift cannot exceed -1. If the domain does not split the drift should be exactly -1 until extinction. So we should expect e.g. a disk domain of area A_0 to vanish under the iteration of this rule in approximately A_0 steps.

Our claim is that given a convex light domain B_0 the average value of the quantity $\sum e_j$ during the iteration of a p-voter c.a. is essentially independent of p and hence positive for all times (upto possible extinction) for all $p \geq 0$. Therefore by the product form of (2.2) the rules with $p < 1/4$ should show annealing/clustering behavior and rules with $p > 1/4$ domain expansion/disintegration. We briefly indicate the reasoning behind this claim.

As the quantity $\sum e_j$ is originally positive consider the two principal mechanisms which can decrease it: (i) The light domain gives birth at an inside corner to a contour inside the original one or (ii) the light domain merges with another one. As there are originally more outside corners than inside ones the event (i) (contributing -2 or -3 if cross points are involved and -4 if not) is on the average countered by at least as many births to the outside each contributing the same but positive amount to $\sum e_j$. So the net effect of the domain births of the type (i) should increase $\sum e_j$. For (ii) we notice that as a consequence of the birth mechanism the finite dark domains are clustered tighter as the light domains since the latter will advance to the exterior of ∂B_0. Hence dark domains are more likely to merge, i.e. the total contribution to $\sum e_j$ is positive. We are here neglecting rare events like the domain folding on itself as we believe that they do not contribute to the expectation in a significant way.

We also note the following implication of (2.2) in the case $p < 1/4$. Given a large disk of radius R then (2.2) and the argument above imply that $2\pi R\mathbf{E}(\Delta R) \approx \mathbf{E}(\Delta A) = c$ for some negative c. So $\mathbf{E}(\Delta R) \propto 1/R$ i.e. the domain shrinkage is governed by the curvature of its boundary. This coincides to the surface tension formulation of contour dynamics in statistical mechanics.

Finally we note that analogous rules and results can be derived in the case of an underlying triangular lattice. Indeed it is a simpler set-up since the even assignment case is absent. For the triangular lattice there is no equivalent balanced rule like the $1/4$-voter c.a. in the case of two symbols per subalphabet but for three there is. The critical probability separating annealing from growth is $1/3$ by the same argument as in Theorem 2.5.

2.3. *Implementation and global dynamics*

The dynamics was also studied in a series of computer simulations. Only small scale runs were possible with the tools available but it must be emphasized that the c.a. considered are computationally extremely efficient. In an ideal situation one would use a parallel computer and a suitable programming language that can match the intrinsic parallelism of the c.a. to the hardware. For us the simulations were just to confirm the design principles of the automata and screen for unexpected behavior.

Since only finite configurations can be simulated the boundary conditions have to be decided. There are at least three different possibilities.

(i) Toral boundary conditions for symbols.
(ii) Toral boundary conditions for the subalphabet assignments and random updates for the digits on the boundary.
(iii) Random boundary updates for the symbols on the boundary.

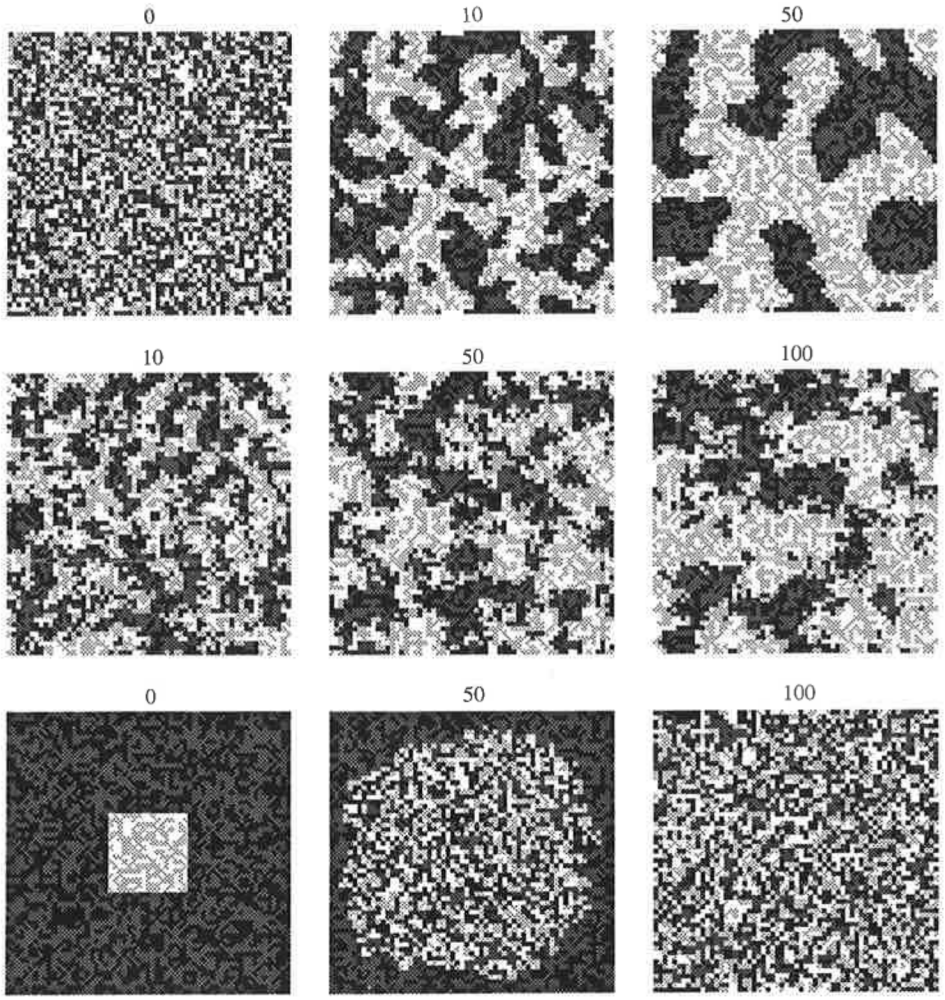

Fig. 2. Top row: the majority voter rule, fully disordered initial state. Middle row: the 1/4-voter rule from the top left initial state. Bottom row: the 1/2-voter rule. All with toral boundary condition on symbols.

In the first choice (i) all pseudorandomness is contained in the initial state. For this reason dependencies might become a dominant feature of the dynamics in long runs. However problems only seem to appear if q^k is a large divisor of the sidelength of the torus (recall that q is the number of symbols per subalphabet). The reason for this is that then large cancellations occur from time to time in the digit evolution. These in turn are a consequence or the digit evolution being determined by an additive action and that the Pascal's pyramid modulo q wraps around the torus in a way which maximizes the cancellations. The existence and analysis of spurious conservation laws in our c.a. is argued analogously to the one-dimensional case ([4]).

One should note and utilize the following feature of the design. The refreshing of the underlying digit evolution at any iterate by replacing the digits with a μ_D-distributed random sample does in no adverse way affect the contour dynamics. Indeed it just breaks any dependencies in the digit evolution by introducing a renewal epoch.

The boundary conditions (i) and (ii) have the advantage that the assignment evolution can stabilize since there is no boundary excitation. In practice the effect of the boundary fluctuations of the assignments seemed

negligible for all p-voter rules with $0 \leq p < 1/4$ (on the square lattice). They all show clear domain separation dynamics resembling surface tension effect. The clustering/nucleation rate increases with decreasing p. As expected the majority voter rule has the cleanest evolutions. In the top row of Fig. 2 we have a sample of this on a 60×60 torus with boundary conditions of the type (i). The initial state at the left corner is uniform Bernoulli on the four symbols. The subalphabets are dark and light and the individual symbols within a subalphabet can be detected from their shade difference. The state after 10 and 50 iterates are shown in the center and at the right. The domain smoothing argued in Section 2.2 is pronounced and the actual "shaving" performed by the merging boundary walks could be seen in consecutive frames.

A series of runs was also performed starting from a disk-shaped domain surrounded by the other subalphabet. The results indicate that for the majority and $1/16$-voter rule c.a. the formula for the expected extinction time behaves like $t^* = A_0/(1 - 4p)$ as suggested the formula (2.2). In the $1/8$-voter c.a. the splintering of the domain is faster and the average value of the critical quantity $\sum e_j$ over the run seems to be at least 5 for the same initial state.

In the middle row of Fig. 2 we have a 100 step evolution of the $1/4$-voter rule from the same initial state (on the top left). The weak clustering expected from the martingale property is clearly visible.

In this context we also note the following on p-values. In our set-up two neighboring cells have overlapping neighborhoods. In updating the cells this causes a dependency as some of the digits are joint. As a consequence of this the true p value is not exactly the one indicated in Table 1. However we did not discover any anomalous behavior as consequence of this effect. In particular the value $1/4$ is the boundary between annealing and diffusive behavior.

No indication was found for the average isotropy of the rules to be distinguishable from true isotropy in evolutions from any initial domain geometry.

2.4. *Point defects*

The contours defined as boundaries between domains generated from different subalphabets are examples of codimension 1 defects. For three-dimensional c.a. they would be random surfaces and we would generate them much the same way as contours.

If three invariant subalphabets are present in a two-dimensional configuration the **triple-points** on domain boundaries where the three subalphabets meet are codimension 2 defects. Like the boundary defects of the majority voter rule they are examples of **point defects.** Let us be a little bit more careful about the definition. A 2×2-neighborhood is said to contain a point defect if its assignment vector $\mathbf{a}$ is of the form (a_0, a_0, a_1, a_2) or its rotation for distinct $a_i \in \{0, 1, 2\}$. If the assignment vector is of the form (a_0, a_1, a_0, a_2) or its rotation for distinct a_i in accordance with our earlier convention we consider four separate domains to be present in the neighborhood. Hence there is no triple point in this neighborhood.

The voter rules can be simply refined to accommodate 2-1-1-divided neighborhoods. The principles listed in Section 2.1 can still be applied to define these plurality voter models ($p \geq 3$, $q \geq 2$). The assignment symmetry now means invariance under a cyclic permutation of the assignments.

From the simple topology of the contours we see that the triple points form an ensemble of pairwise annihilating random walks. They are either originally present or are born in pairs as a consequence of a contour joining another contour. It is notable that even in the simplest of rules, the extended majority voter rule, the births are already present and the point defects can jump arbitrary long distances in one iterate.

2.5. Diffusive domains

After analyzing the annealing p-voter rules it is quite natural to inquire about the dynamics of the rules with $p > 1/4$. Some of the p-values available in this interval are listed in Table 1 of Section 2.1 and the rest are $1-p$ as p runs through the given ones. For $p \geq 1/2$ the majority domination assumption (ii) in the beginning of Section 2.1 is violated.

The dominant dynamics in the case $p = 1$ is immediate: the corner cells of a domain advance at maximal rate $(\pm 1/2, \pm 1/2)$ into the neighboring domain and consequently every domain is bound to disintegrate. As in the case of annealing voter rules this behavior prevails in a stochastic form if exceptions dependent on the digits are allowed, i.e. $1/4 < p < 1$. All these p-voter c.a. exhibit diffusive behavior as predicted in Section 2.2 and the density of the subalphabets in the interior approaches quickly $1/2$. The time scale of the disintegration of domains varies monotonically in p.

At the bottom row of Fig. 2 we have a generic sample of this behavior at $p = 1/2$ using a rule with same symbols as in the rules in Section 2.3 and average isotropy. The lattice size is 60×60 and the boundary condition is periodic on symbols. Experiments with other diffusive rules upto around p value $3/4$ yielded similar evolutions. Phenomena encountered in this growth regime include a second critical probability and are studied in more detail in [6] using the the independent model.

3. Critical phenomena

The p-voter c.a. can be further extended to accommodate interesting new phenomena. In this section we will introduce a second parameter physically analogous to temperature for which a critical value exists indicating the fact that the c.a. family experiences a phase transition.

The splitting of the c.a. rule introduced in (1.1) gives us a degree of freedom that goes beyond the subalphabet invariance. The fact that

$$A(\mathbf{a}, \mathbf{d}) \equiv a \quad \text{whenever } \mathbf{a} = (a, a, a, a)\,, \tag{3.1}$$

together with a quasigroup action on the digits implies that the block map f permutes the subalphabet symbols and we have the subalphabet invariance of Definition 1.2. Note however that the critical stationarity property, i.e. preservation of a measure of the type indicated in (1.2), is preserved even if we violate (3.1). If we do this **spontaneous births** are introduced i.e. a 2×2 neighborhood with four symbols from the same subalphabet can generate a symbol belonging to the other one. The physical idea of (3.1) is that of zero temperature, i.e. state where no spontaneous flips occur in an homogeneous medium. We will now break that without changing the action on split neighborhoods and consequently the contour dynamics will remain the same.

A ***b*-*p*-voter rule** is defined from the p-voter rule of Section 2.1 by allowing the invariance (3.1) to be broken with probability b. In 3-1 and 2-2 split neighborhoods the rule is exactly the p-voter rule. The triangular lattice case is modified analogously, i.e. just 3-0 neighborhood action is changed. We only consider the case of two subalphabets here. The invariance (3.1) is broken with the same mechanism as in p-voter rules,

$$\text{(3.1) is violated if } \sum_{i=1}^{4} d_i \in B_b\,,$$

where B_b is the set of exceptions for births. The possibilities for the sets B_b and values b are determined as for C_p and p at the end of Section 2.1 (but the sets B_b and C_p are in general different). Clearly $B_0 = \emptyset$ gives

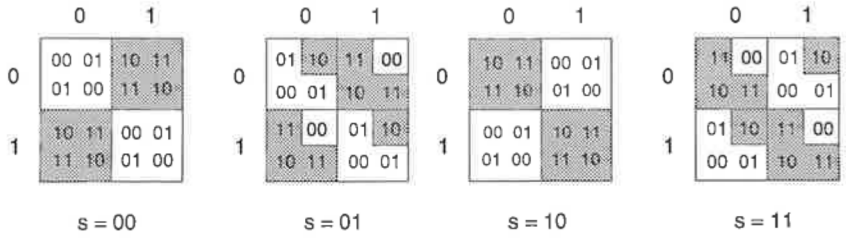

Fig. 3. The $(1/8, 7/8)$-voter rule: a quasigroup c.a. on the triangular lattice.

us the p-voter rules.

Apart from being functionally natural the b-p-parametrization turns out to have appealing algebraic structure as well. There is a complementarity relation between the parameters which we now describe. Given the digit quasigroup and hence the allowed b- and p-values let us call the set of all c.a. that arise from them as the set of b-p-rules and use the notation $F_{(b,p)}$ for the global c.a. map.

By contour dynamics we mean the motion and interaction of the domain boundaries. This determines the essential nature of the rule as e.g. in the context of Ising model. The following result only depend on the assignments so it holds to c.a. as well as to their independent models.

Proposition 3.1. *The set of b-p-rules on either lattice are functionally symmetric with respect to the point $P = (b, p) = (1/2, 1/2)$ in the sense that for every c.a. $F_{(b,p)}$ there is a c.a. $F_{(1-b,1-p)}$ with exactly the same contour dynamics.*

Remark. Note that the results only addresses the contour dynamics and not the domain dynamics. The latter is not identical for the complementary rule at odd times as indicated in the proof. An immediate practical implication of the result is of course that only half of the rule space needs to be investigated.

To argue the phase portrait we now consider the invariant measures for the b-p-voter rules. The running assumption (1.2) still guarantees maximally mixing digit dynamics.

From (i) of Section 2.1 we know that p-voter rules preserve the measures μ with constant assignment: $\pi_1 \mu = \delta_{\underline{0}}$ or $\delta_{\underline{1}}$ (the pure phases). Above the critical probability p_c a third invariant measure appears. This disordered phase dominates the homogeneous phases and growth phenomena ensue as indicated in Section 2.5. For further details of this see [6].

The third phase which always seems unique, translation invariant and disordered can be in some cases be exactly pinned down.

Proposition 3.2. *All the rules which are quasigroups, i.e. permutive on the entire alphabet S, are on the diagonal $p = 1 - b$. In the square lattice case a quasigroup rule must be at P. A quasigroup rule preserves the uniform Bernoulli measure on symbols.*

Remark. In the case of the triangular lattice there are indeed quasigroups on the diagonal $p = 1 - b$ outside P. In Fig. 3 we have the multiplication table of a $(1/8, 7/8)$-voter rule. The subalphabets are $S^{(0)} = \{00, 01\}$ and $S^{(1)} = \{10, 11\}$. The four slices should be stacked from left to right to get the "multiplication cube". On the top and left of each slice we have indicated the subalphabet assignments and the convention white for 0 and grey for 1 assignment has been used to make the structure of the cube easier to grasp. Note that by flipping the assignments at the eight framed 1×1 squares the table turns into a rule with parameters $(b, p) = (0, 1)$ which

is an example of a diffusive c.a. with extremely unstable domain behavior in the sense discussed in Section 2.5 (a 1-voter rule).

For the independent model of a b-p-voter rule this result can be sharpened to characterize when the disordered phase is simple.

Theorem 3.3. *The independent b-p-voter c.a. on the triangular lattice preserves the uniform Bernoulli measure iff $p = 1 - b$. On the square lattice the measure is preserved iff $2b + 4p = 3$.*

The last statements give us an indication how the rules are ordered in the parameter space $[0,1]^2$ in terms of their mixing properties: as quasigroup rules are fully permutive the maximal degree of mixing (i.e. "chaos") is found along the lines $p = 1 - b$ and $p = (3 - 2b)/4$ depending on the lattice. We will see shortly that the least degree of mixing is found furthest away from these, i.e. at the corners $(b, p) = (0,0)$ and $(1,1)$.

Before considering ergodicity one has to acknowledge the existence of spurious invariant measures. By these we mean measures that do not appear to correspond to any physical phenomenon but are rather combinatorial artifacts. Due to their extreme instability under perturbations they are not encountered in probabilistic models. The only ones that we found were for b-0-voter rules. One constructs such an object by simply extending 0011 to a bi-infinite strip tilted 45 degrees either way and then periodically continuing the strip to the square lattice. The resulting configuration is invariant (modulo a shift) and thereby gives rise to a "exotic" invariant measure. An even simpler construction can be performed for the triangular lattice. However these measures are highly unstable under flip perturbations and they do not persist for positive p. We exclude these measures from further consideration.

In the parameter regime of perhaps most interest $0 \leq p < 1/4$ and $0 < b < 1/2$ our rules have two conflicting tendencies: the annealing behavior analyzed earlier which tends to shrink finite domains and the births which create these. This structure is indeed quite close to the one found in dynamic Ising models. These have a critical parameter value and it should exist in our models for the given p-values as well.

A b-p-rule is said to be **ergodic** if for all reasonable initial measures μ we have $F^{2n}_{(b,p)}\mu \Rightarrow \mu^*$ where μ^* is a unique equilibrium measure and $\Rightarrow$ denotes the weak convergence of probability measures. If the uniqueness fails the c.a. is non-ergodic. The choice of the set of allowed initial measures must be made with care in the context of deterministic c.a. This is necessary since if for example measures concentrated on periodic points were allowed the ergodicity could be easily contradicted for almost any rule (a c.a. has typically multiple invariant measures supported by periodic points, see [3]). In our context it seems natural to inquire about ergodicity when the initial measure satisfies the stationarity condition (1.2) and the exotic invariant measures are excluded.

A p-voter rule for any p is non-ergodic since any initial measure μ supported by configurations from one of the subalphabets only remains such under the rule. To decide the ergodicity of the b-p-rules, $b > 0$, we argue as follows. If there is a unique equilibrium measure μ^* then the density of subalphabet $S^{(1)}$ in a generic configuration (with respect to μ^*) must equal to $\rho^* = 1/2$. This follows from the assignment symmetry of the subalphabets. If the equilibrium is not unique there will be conjugate (under the flip) equilibrium measures such that for their generic configurations the density of $S^{(1)}$ satisfies $\rho^*_+ > 1/2$, $\rho^*_- < 1/2$ and $\rho^*_+ + \rho^*_- = 1$. If our initial measure has $S^{(1)}$ density zero it seems reasonable to assume that the density of a generic point evolves to the minimal ρ^*_-. This is just because arriving at the other equilibrium requires a large deviation event. So to test whether we have non-uniqueness of the equilibrium measure it suffices to check whether the density of the subalphabet $S^{(1)}$ initially absent approaches a value strictly below $1/2$.

Fig. 4b illustrates the results from a series of runs with different b- and p-values on a square lattice. Most of the runs were performed on a 100×100 table with boundary condition of the type (ii) in Section 2.3. Near the

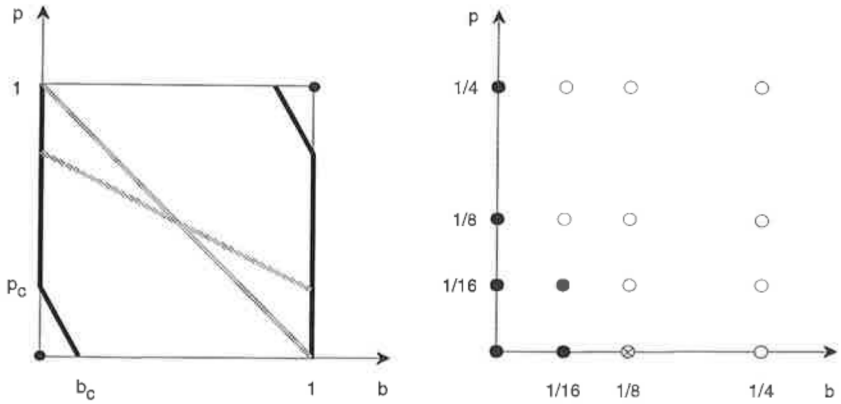

Fig. 4. Phase diagram of the b-p-voter rules and its critical corner.

critical parameter pairs (b, p) the table was extended up to size 170×170. The boundary condition is here of paramount importance – in the neighborhood of the critical parameter value the influence of the boundary could dominate the evolution as a consequence of the increasing length scale. The black dots indicate non-ergodic behavior and white ones ergodic. The 1/16-1/16-voter rule is quite close to ergodic with ρ^* between 0.4 and 0.44 but multiple runs with consistent results convinced us that it is indeed non-ergodic. The 1/8-0-rule is even closer to critical – it is likely to be ergodic but we do not have enough data to resolve this (hence $\otimes$ in the graph).

The non-ergodicity in the triangular lattice case is expected to be analogous. It should be noted that the independent model there is a symmetrized Toom's model. For the original model the phase transition has been proved to exist ([13]) and this property should carry over to our set-up. For numerical estimates and some statistical mechanical insight to the original model, see e.g. [10].

The figure on the left summarizes our findings concerning the phase diagrams. By Proposition 3.1 it has central symmetry. The wedge-like corners and the black lines bounding them accommodate the non-ergodic rules. The dots at the majority voter rules refer to the annealing result expressed in Proposition 2.6. Between the black curves (which should be characterized more accurately in numerical or other studies) is the disordered regime. The grey lines are as in Proposition 3.2 and Theorem 3.3 indicating the preservation of the uniform Bernoulli measure. The critical probabilities are dependent on the lattice but we strongly believe that the phase portrait is qualitatively as shown for both of them. More importantly in view of our results the phase diagram is essentially the same for the b-p-voter rules and their independent models (apart from the obvious discretization of parameters).

4. Conclusion

We have presented a family of two-dimensional c.a. rules which exhibit a variety of different type of domain dynamics. The principles according to which the rules have been built guarantee that the evolutions resulting are maximally random in the context of deterministic c.a. Deterministic rules are in turn our choice since (i) they are computationally superior and therefore perhaps useful in large scale physics computations and (ii) provide a challenge to explain why the randomness can indeed prevail is such striking fashion in them.

The qualitative properties of the rules seem to be continuously dependent on the two parameters except for the phase transition. No surprising conservation laws which would render the rules useless in the simulation of

statistical physical processes are found.

Finally we note that the rules given (the assignment structure in particular) can be modified in a number of ways. For example the available parameter ranges can be extended by alternating two rules with slightly different parameters. This has not been tested but we do not see any problems arising especially if the resulting dynamics is sampled by averaging, e.g. over two iterates. Also the principles in Section 2.1 could be further modified. For example removing the unbiasedness implemented in (iii) and (v) would bring out novel wavefront propagation phenomena while still preserving the stationarity.

Acknowledgement

The author thanks Eugene Speer for discussions on Toom's model and the anonymous referee for useful comments.

Appendix

As for Theorem 1.4 we only indicate here the details in the set-up which facilitate the use of more general results recorded in [8]. The proof in our set-up of rectangular lattice hinges on the fact that the polynomial $1 + x + y + z + xy$ is irreducible over the finite field $\mathbf{Z}_2$ (recall that our quasigroups have mod 2 addition). The polynomial in turn is determined as a dual representation in the polynomial ring $\mathbf{Z}/2\mathbf{Z}\left[x^{\pm}, y^{\pm}, z^{\pm}\right]$ of the shifts appearing in the definition of the subshift $D^{(3)}$. The unit shifts σ_h, σ_v and σ_t correspond to multiplication by x, y and z, respectively, in the polynomial ring. Hence the geometric arrangement of these unit shifts as vertices of a pyramid gives the polynomials (see Fig. 1 neighborhoods: as time runs perpendicular to the spatial directions the black cells should be thought to lie above the neighborhood).

Proof of Theorem 2.4. Let L be the NW face (hull line) of the confining diamond. On L we can identify the highest and lowest points where the hull line touches the light domain (suppose the island is of the light subalphabet). Denote these top and bottom corners on the face at time i by T_i and B_i. Under the convexity assumption the variables are always well defined even if the domain splits. Let d_i be the distance of the top and bottom corner along the face. T_i and B_i jump up and down along the face until they coalesce or annihilate (then $d_i = 0$). Assume first that $T_i \neq B_i$ and consider the motion of T_i. Two possibilities arise which are illustrated in Fig. 5.

If T_i is a corner of a cell "sticking out" as case a in the figure then $\Pr(T_i \text{ jumps up} \mid T_i \neq B_i) = 0$. If case b arises then the location of T_{i+1} depends on whether the update of the gray cell on top of the 2-2-split neighborhood is light or dark. But the grey cell is determined from the backward pyramid whose top it is. If the grey cell is at the origin at time i the corners of the base of the pyramid are at $(\pm i/2, \pm i/2)$. By the definition of the majority voter rule the digits at these corners permute the digit of the grey cell. Hence by the assumption (1.2) the update is light or dark with equal probabilities and $\Pr(T_i \text{ jumps up} \mid T_i \neq B_i) = 1/2$. Clearly T_i cannot stay put. If it does not jump up it either jumps down or annihilates with B_i (and exits L, i.e. the face, moves in). Hence $\Pr(T_i \text{ jumps down or exits } L \mid T_i \neq B_i) \geq 1/2$. The motion of B_i is argued identically.

The motions of T_i and B_i are independent. When the jump isn't forced (case a) the grey cells for T_i and B_i are distinct. Their updates are determined permutively in particular from the NW corners of the bases of their respective backward pyramids. For disjoint grey cells these corners are always disjoint and by (1.2) their digits are independent. Hence $\Pr(d_{i+1} < d_i \mid T_i \neq B_i) \geq 1/4$. By Theorem 2.1 d_i is bounded from above. So

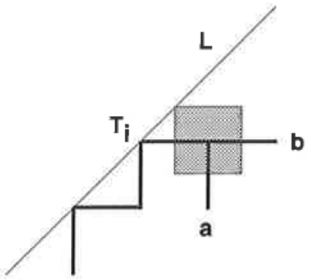

Fig. 5. Jump of the top corner on the hull line.

Pr ($d_i = 0$ for a finite i) = 1. If $d_i = 0$ the corners either have coalesced or annihilated each other. In the case of coalescing the face L moves in, i.e. the hull shrinks in the next iterate with probability 1/4.

Unless the domain becomes extinct after the annihilation we redefine T_0 and B_0 on the new NW hull line and iterate the argument above. By the finiteness of the island and Theorem 2.1 this needs to be repeated only a finite number of times. Note that after a finite time interval the NW corners of the new backward pyramids extend further to that direction than any previous ones. Therefore from then on the jumps are again independent of the past. □

Proof of Theorem 2.5. Let us first consider the case where there is just one (outside boundary) component in ∂B_j. Consider the set of all 2×2-neighborhoods through which the boundary contour travels (i.e. splits them evenly or divides them in 3-1). For any bounded domain this set is finite. In any neighborhood the cell in the dual lattice at the center as in Fig. 1 is to be determined in the next iterate. In the neighborhoods where the boundary is straight or there is a cross point the expected domain gain in one iteration is zero. At corners the cell is part of a 3-1-arrangement. By the way the probabilities are determined in a p-voter rule the expected domain area change at an outside corner is $\Delta A = p\,\frac{3}{4} - (1-p)\frac{1}{4} = -(1-4p)/4$. At an inside corner the area contribution is $-\Delta A$. So the total contribution from the contour is $\Delta A e\left(\partial B_j\right)$.

If the domain is not simply connected the contribution from the interior boundary is still computed by recording the excess along the curve and by our choice of the traverse direction the contribution is $\Delta A e\left(\partial B_{\text{int}}\right)$. Adding these contributions over all components yields (2.2). Note that as two contours are at least distance one apart except at neighborhoods where there is a cross point their contributions to the total shrinkage are actually independent. □

Proof of Proposition 2.6. If a domain has no cross points and all its n components are simply connected then $\sum e_j = 4n$. Two nested contours that bound the same domain and do not have cross points have opposite excesses. So if there are no cross points but contours are allowed to be nested also non-positive multiples of 4 are possible for $\sum e_j$.

It is an immediate consequence of the convex confinement argument formulated in Theorem 2.1 that the majority voter rule cannot form holes in a domain initially convex. However it can split a domain into convex pieces. If their diamond hulls intersect there can be a cross point in which case $\sum e_j$ does not need to be a multiple of 4. But in this case the value of the expression is at least 6.

For the last statement we notice that as $A_i = 0$ is an absorbing state,

$$\begin{aligned}\mathbf{E}\left[A_{i+1}\right] &= \Pr\left(A_i > 0\right)\mathbf{E}\left[A_{i+1}|\, A_i > 0\right] + \Pr\left(A_i = 0\right)\mathbf{E}\left[A_{i+1}|\, A_i = 0\right] \\ &= \Pr\left(A_i > 0\right)\left(\mathbf{E}\left[A_i|\, A_i > 0\right] - \frac{1}{4}\sum e_j\right) \\ &= \mathbf{E}\left[A_i\right] - c\Pr\left(A_i > 0\right)\,,\end{aligned}$$

where $c \geq 1$ for convex B. Therefore $\{\mathbf{E}\left[A_i\right]\}_{i\geq 0}$ is a non-increasing sequence bounded from below. So it has a limit and consequently $\Pr\left(A_i > 0\right) \to 0$. But as 0 absorbs this immediately implies that $\Pr\left(A_i > 0 \text{ i.o.}\right) = \Pr\left(A_i > 0\ \forall i\right) = 0$. □

Proof of Proposition 3.1. The symmetry with respect to P is a consequence of the assignment symmetry. Recall that the map ι defined by $\iota((a,d)) = (1-a,d)$ commutes with the rule of the c.a., i.e. $f \circ \iota = \iota \circ f$, where it is understood that ι as an argument in f is applied to all argument assignments (three or four depending on the lattice). This conjugation extends to a global map: $F \circ I = I \circ F$, where I flips the assignment at every lattice site.

Given a b-p-voter c.a. we can define the complementary rule simply by $F_{(1-b,1-p)} = F_{(b,p)} \circ I = I \circ F_{(b,p)}$. Applying the c.a. $F_{(b,p)}$ and $F_{(1-b,1-p)}$ to an arbitrary configuration moves any existing contours exactly the same way. The updated assignments of the cells in the interiors of the domains are reversed but this does not affect the contour motion. Moreover by using the assignment symmetry we see that even this is removed in second iterate since $F^2_{(1-b,1-p)} = \left(F_{(b,p)} \circ I\right)^2 = F_{(b,p)} \circ I^2 \circ F_{(b,p)} = F^2_{(b,p)}$. □

Proof of Proposition 3.2. To locate the quasigroup rules let us first consider the case of a rule on a triangular lattice. So we need $f(S, s_2, s_3) = S$ for any fixed s_2 and s_3 in S, the full alphabet. In particular it must then hold that when varying s_1 half of the symbols in the image are in $S^{(1)}$ as the subalphabets are of equal size. There are two cases: 1. $s_2 = s_3 = 0$. As s_1 is varied depending on whether it is in $S^{(0)}$ or $S^{(1)}$, the frequency of getting a symbol in $S^{(1)}$ is either b or p. Therefore it must hold that $(b+p)/2 = 1/2$ by the even representation of the subalphabets. So $p = 1 - b$. 2. If $s_2 = 0$ and $s_3 = 1$ the corresponding probabilities are p and $1 - p$ hence the equation for even representation in the image under f is $(p + 1 - p)/2 = 1/2$, an identity. The other choices of s_2 and s_3 reduce to these.

In the case of square lattice one proceeds the same way but now the novelty is the case of even representation of the subalphabets in the neighborhood. If $(s_2, s_3, s_4) = (0,0,1)$ or $(0,1,1)$ or any of their cyclic permutations the frequency of the image $f(s_1, s_2, s_3, s_4)$ being in $S^{(1)}$ as s_1 ranges through S is $(p + 1/2)/2$. As this has to equal $1/2$ the value of p is fixed at $1/2$. The other case gives again $b = 1 - p$.

By Section 1 a quasigroup c.a. preserves the uniform Bernoulli measure. □

Sketch of the proof of Theorem 3.3. We only indicate the key step in the square lattice case here.

To locate the candidates for Bernoulli-preserving b-p-voter rules one considers a 2×3 neighborhood and calculates the probability of a second 0 update given one 0 update. By enumerating the different cases one arrives at the expression

$$\Pr(0|0) = \frac{1}{32}\left(\frac{41}{2} - 6b + 2b^2 + 8bp - 12p + 8p^2\right).$$

A necessary condition for the uniform Bernoulli measure to be preserved is that this equals to $1/2$. The line $2b + 4p = 3$ is the double root for the equation.

On the candidate line one then checks that if the initial distribution is uniformly Bernoulli then the distribution of any one or two cells in the 2×2 neighborhood is independent of the update. Using this independence property

on the parameter line the extension to an arbitrary cylinderset is straightforward. For the details of this and for the analogous triangular lattice case we refer to [6]. □

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