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Critical growth phenomena in cellular automata

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Abstract

We study a one-parameter family of probabilistic cellular automata on square and triangular lattices. Above a critical parameter value a new dominant invariant phase appears resulting in domain growth. In the growth regime a second critical threshold is found above which domains grow at a maximal rate (and facet). This phenomenon is shown to be equivalent to a certain one-dimensional directed percolation problem studied by Domany and Kinzel (1984). Tight bounds are given to the critical probabilities which depend on the lattice. These models are of special interest since their behavior corresponds extremely closely to that of certain simple purely deterministic cellular automata.

Keywords: Cellular automaton; Oriented percolaton; Growth model; Faceting

0. Introduction

Growth processes arise as simplified models to a diverse range of phenomena from crystal formation to population expansion. Broadly speaking they can be viewed as an interaction of two phases/species with the distinctive feature that one of them is dominant and thereby forms expanding domains. How exactly this expansion takes place can depend in a delicate way from the local characteristics of the rule as well as the underlying structure (lattice, etc.). Resolving these questions has received a fair amount of attention lately, see e.g. [1,6].

Our objective here is to investigate certain probabilistic growth models on planar lattices. Although the models have an intrinsic appeal due to their simplicity the original motivation comes from the dynamics of deterministic cellular automata [4]. Indeed the parallel between these models is very close and our claim is that the results here apply almost verbatim to the original context. We restrict to dynamics on two-dimensional lattices but the principles extend to higher dimensions as well (and one is trivial).

The models that we consider are a one-parameter family of nearest neighbor voter-type rules. These are set up in Section 1. After that we give an outline of the dynamics indicating the growth regime. For most of Section 2 we concentrate on growth phenomena resulting in faceting. By indicating the appropriate percolation formulation the faceting is then shown to happen for certain parameter values. We also give tight numerical bounds for the critical threshold values. A partial characterization of the dominant phase is formulated. Finally we briefly note the connection to deterministic cellular automata dynamics.

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1. Lattices and rules

Let S_e be the square lattice $\mathbf{Z}^2$ and S_o its dual lattice $\mathbf{Z}^2+(\frac{1}{2},\frac{1}{2})$. Denote by T_e the triangular lattice oriented in such a way that origin is a lattice point and one of its six nearest neighbors is at $(1,0)$. Let $T_o = T_e + (0, \frac{1}{3}\sqrt{3})$. T_o should be thought as a $\frac{1}{2}$-thinning of the hexagonal lattice (which is the dual lattice of the triangular lattice). S_e is generated by $\{(1,0),(0,1)\}$ and T_e by $\{(1,0),(\frac{1}{2}\sqrt{3})\}$.

Our two states/species are denoted by the symbols 0 and 1. Given any of the four lattices L the set $X = \{0,1\}^L$ is a set of configurations. On this we have the natural coordinate actions, the horizontal and vertical shifts defined by the generators.

In the case of a square lattice we consider iteration of map on configurations $F:\{0,1\}^{S_e} \to \{0,1\}^{S_o}$ and $F:\{0,1\}^{S_o} \to \{0,1\}^{S_e}$ given by a local rule f. This is a mapping $f:\{0,1\}^4 \to \{0,1\}$ on a 2×2 neighborhood which determines the value at the center site belonging to the dual lattice. F is obtained by applying the rule in every neighborhood simultaneously, so it commutes with the coordinate shifts. In the triangular lattice case two maps alternate: $F_e:\{0,1\}^{S_e} \to \{0,1\}^{S_o}$ and $F_o:\{0,1\}^{S_o} \to \{0,1\}^{S_e}$. Both the maps are given up to orientation by the same rule on a neighborhood triple.

In Fig. 1 the setups are illustrated: (a) and (d) are the lattice arrangements and (c), (e) and (f) are the neighborhoods. Fig. 1(b) illustrates the square neighborhood and the update with unit cells.

The reason for studying models in this alternating lattice setup is twofold. First it provides a simple two-dimensional neighborhood structure which is nevertheless quite general: any finite-to-one map f on a square or triangular lattice can be transformed to the square case (with perhaps more states). As the neighborhood in the triangular case is even more elementary we consider it for the sake of comparison and it indeed reveals some qualitatively different features. The price for the simplicity there is the loss of isotropy which has to be made up by alternating rules oriented in opposite ways. Second the motivation comes from the analysis of pseudo-random dynamics in deterministic cellular automata. There the key ingredient, partial permutivity, can only be defined if the lattice-dual lattice setup is used [4]. It is of interest to see how a qualitatively similar rule behaves when this feature is upgraded to independence as in this study.

To complete the preliminaries we have to define the local rules. An n-m-distributed neighborhood has n 0's and m 1's. A homogeneous neighborhood is all-zero or all-one and an uneven is 2-1 or 3-1 and an even neighborhood is 2-2 distributed in any fashion.

Definition 1.1. A *p-voter rule* f on a square or a triangular neighborhood maps 4-0 (0-4) or 3-0 (0-3) neighborhoods to 0 (1) with probability (w.p.) 1 and 2-1 (1-2) or 3-1 (1-3) neighborhoods to 1 (0) w.p. p. In the 2-2 neighborhoods we get 0 or 1 with probability $\frac{1}{2}$. The outcomes from disjoint neighborhoods are independent.

Remarks.
1. The rules simply implement an isotropic voter behavior. The parameter p measures the chances of the minority to prevail. Its inverse can be thought to reflect co-operation among the majority.
2. The rules are perfectly symmetric with respect to the states. Moreover both homogeneous configurations are invariant. Hence the measures $\delta_{\underline{0}}$ and $\delta_{\underline{1}}$ on the appropriate lattices are preserved by F, F_e and F_o and the rules are nonergodic for all p. But there are other translation invariant measures as we will see in the next section.

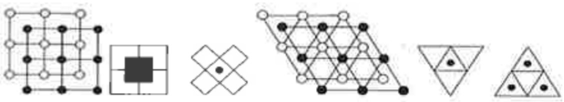

Fig. 1. (a) Square lattice and its dual. (b) The cell neighborhood and update. (c) Diamond neighborhood. (d) Triangular lattices. (e), (f) Triangle neighborhoods for even and odd iterates.

2. Dynamics

The dynamics of the p-voter rules divides into three distinct regimes in both triangular and square lattice case. Between them there are two critical probabilities which are dependent on the lattice. In this section we elaborate these phase diagrams and in particular the upper critical values. The low end of the parameter range was studied in [3,4] so we only briefly summarize it here.

Below the lower critical value $p_s^g = \frac{1}{4}$ in the square lattice case and $p_t^g = \frac{1}{3}$ in the triangular lattice case the evolution from a disordered state seems to lead to phase separation. More specifically we expect the weak convergence

$$F^i \mu \Rightarrow \lambda \delta_{\underline{0}} + (1-\lambda)\delta_{\underline{1}}, \quad 0 \le \lambda \le 1.$$

Here μ is any initial distribution of 0's and 1's. Although there seems to be very little doubt about this happening, there is no proof. As a partial remedy we know rigorously that for $p = 0$ finite islands will vanish almost surely [4].

The conjectured critical values can be justified as follows. By considering the expected area of a single finite island D of one phase surrounded by an infinite sea of the other one arrives at the formula (again see [4])

$$\boldsymbol{E}[A_{i+1} | A_i > 0] = A_i + \frac{kp-1}{k} e_{\partial D_i}. \tag{2.1}$$

Here A_i is the area of D at time i and k is either 4 or 3 depending on whether the square or triangular case is considered. The quantity $e_{\partial D_i}$ is a certain simply computable index of the boundary of D_i. It is integer-valued and for almost all domain geometries positive. Hence p seems to determine the sign of the drift of the area process and the given critical values p^g are the values at which the drift vanishes; and indeed simulating the rules indicates nothing to refute this.

2.1. Growth

By (2.1) above p^g. the p-voter rules should be expected to increase the area of a finite domain and thereby serve as growth models (hence the superindex g). In this section we characterize the growth and the asymptotic shape of an expanding domain.

To provide some intuition Fig. 2 illustrates the outcomes of simulations in the square lattice case where a seed entry 1 was placed at origin while the rest of the lattice is all 0's. 1 is rendered as a black and 0 as a white square cell and the cells fill up the lattice as shown in Fig. 1(b). The figures are results from 200 iterate runs with parameter values $p = \frac{1}{2}, \frac{3}{4}$ and $\frac{7}{8}$. Note that the frame does not represent a boundary condition – there is none. It just indicate the maximal size to which the domain could have grown in about 220 steps (and thereby shows the common time scale of the three samples).

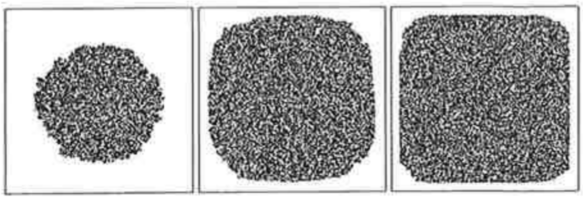

Fig. 2. Domain after 200 iterates from a single black cell in the center, $p = \frac{1}{2}, \frac{3}{4}$ and $\frac{7}{8}$.

Well below the second critical level p_s^e the shape of the expanding domain is roundish, its boundary rough and the underlying lattice does not immediately reveal its orientation. Approaching the critical value the boundary squares off and above it the domain facets, i.e. it has flat edges whose lengths are proportional to their distance from the origin. Faceting is a result of the front cells advancing at the maximal rate. Fig. 2(b) illustrates the behavior very close to the critical probability. The faceting is monotone in the sense that the asymptotic shape is increasingly squarish being exactly a square at the limit $p = 1$.

An analogous phenomenon takes place in the p-voter rules on the triangular lattice. The upper critical value p_t^e is different though and so is the asymptotic shape. The expanding domain tends to a hexagon with the faces oriented parallel to the lattice axes.

2.2. Percolation formulation

We now proceed to analyze the growth and to estimate the upper critical values. One should also note that this result will of course imply the existence of a third invariant measure which assigns nontrivial probabilities to both symbols.

Let us first consider the square lattice case. The local dynamics is easiest to describe when one imagines a diamond of sidelength $\frac{1}{2}\sqrt{2}$ drawn around every lattice point as in Fig. 1(c) and then superimposes the arrangements on the two lattices. Suppose that the seed diamond at origin is labeled 1 and the rest on the even lattice is 0. Let the positive y-axis be the direction into which we consider the influence of the seed. A p-voter rule turns the diamonds centered at $(\pm\frac{1}{2}, \frac{1}{2})$ to 1's independently w.p. p. In the next iterate the diamonds at $(\pm 1, 1)$ are argued similarly, i.e. turned to 1 w.p. p if the ones at $(\pm\frac{1}{2}, \frac{1}{2})$ were 1's. If the ones at $(\pm\frac{1}{2}, \frac{1}{2})$ were both turned 1's then the diamond at $(0, 1)$ is turned to 1 w.p. $\frac{1}{2}$. If exactly one of the diamonds at $(\pm\frac{1}{2}, \frac{1}{2})$ was 0 then (0,1) is 1 again w.p. p and if neither was 1 the diamond at (0,1) remains 0. In summary the probabilistic rule of assignments in one step to the given direction (now distance $\frac{1}{2}$ upwards) is

$$\begin{array}{ll} (0,0) \to 0 & \text{w.p. } 1, \\ (0,0), (1,0) \to 1 & \text{w.p. } p, \\ (1,1) \to 1 & \text{w.p. } \frac{1}{2}. \end{array} \tag{2.2}$$

Given the arrangement above let F_i be the set of sites on the extreme front, i.e. on the line $y = \frac{1}{2}i$ that turn 1 at the ith iterate (there cannot possibly be any such sites further up). We define that a p-voter rule *exhibits extreme growth in the given direction if* $\Pr(\{F_i \neq \emptyset\ \forall i\} > 0$, i.e. with positive probability there is always a nonempty cluster of extreme front sites. Note that the propagation of the 1's to the direction under consideration can be viewed as flipping the diamonds around the two top edges with the probabilities given in (2.2). If we do not erase the trail of diamonds leading to the front, the growth problem is coverted to a directed percolation problem. F_i is nonempty exactly when there is a diamond trail of 1's of length i from the seed to a diamond on the line.

In the triangular lattice case one needs to keep track of the parity of the lattice but the argument is similar to the square case. Instead of diamonds one now places isosceles triangles on the lattice points. On T_e they stand on a corner, on T_o on an edge and their size such that superimposing the arrangements partitions the plane.

The single seed 1 at the origin is on the even lattice T_e (time zero). Elementary geometry shows that the front line perpendicular to the y-axis must now be

$$y = \begin{cases} \frac{1}{2}\sqrt{3} \cdot \frac{1}{2}i, & i \text{ even}, \\ \frac{1}{2}\sqrt{3} \cdot \frac{1}{2}(i-1) + \frac{1}{3}\sqrt{3}, & i \text{ odd}. \end{cases}$$

Again it is useful to think of the front advancing through flipping of its triangles around those edges yielding a successor with a higher center. A triangle on the even lattice flips in such a fashion with probability p independent of the other triangles on the front. On the odd lattice the flipping rule is as (2.2) but now the p-voter rule tells us that $(1, 1) \to 1$ w.p. $1 - p$.

The process can be further simplified by gluing together two triangles sharing a horizontal edge (the lower to T_e and upper on T_o into a lozenge. Let the lozenge have value 1 if the triangles that constitute it are both 1 and 0 otherwise (0-triangle on top of a 1-triangle or both triangles 0's). On these lozenges we have a probabilistic binary rule much like in the square case:

$$\begin{array}{ll} (0,0) \to 0 & \text{w.p. } 1, \\ (0,1), (1,0) \to 1 & \text{w.p. } p^2, \\ (1,1) \to 1 & \text{w.p. } p(1-p). \end{array} \tag{2.3}$$

Extreme growth is formulated and converted to a directed percolation problem exactly as in the square lattice case. The facets to directions $\{\frac{1}{2}\pi, \frac{1}{2}\pi \pm \frac{2}{3}\pi\}$ are defined as above and to the directions $\{\frac{3}{2}\pi, \frac{3}{2}\pi \pm \frac{2}{3}\pi\}$ from the triangles neighboring the seed. The effect of the resulting off-set on the asymptotic shape is of course negligible (as asymptotic shape is defined as usual via a scaling limit).

In a pair of papers Domany and Kinzel [2] investigated directed percolation in evolutions of a two-parameter family of probabilistic binary cellular automata. The model was simply $(0,0) \to 0$, $(0,1), (1,0) \to 1$ w.p. p_1 and $(1,1) \to 1$ w.p. p_2. Using transfer matrix scaling techniques they calculated among other things the critical directed percolation probabilities $p_1(p_2)$. The accuracy of the phase diagram has been slightly improved in later studies (e.g. [7]).

Our rules (2.2) and (2.3) correspond to the cases $(p_1, p_2) = (p, \frac{1}{2})$ and $(p^2, p(1-p))$. So given the construction above we have established a rigorous relation between the two growth models and the directed percolation model.

Theorem 2.1. Both the growth models exhibit extreme growth above a critical probability. In the square lattice case this threshold is $p_s^e = p_1(\frac{1}{2})$. On the triangular lattice p_t^e is the p_1-coordinate of the intersection of the curves $p_1(p_2)$ and $p_2 = \sqrt{p_1}(1 - \sqrt{p_1})$.

Remarks.

1. Since a partially unknown function $p_1\ (p_2)$ of [2] is involved, a comment is in order. Durrett and others (see e.g. [1]) have rigorized some of the arguments in [2] but not in the most difficult regime $p_1 > p_2$ relevant to our models. In particular it is not known whether the critical probabilities are below one. However there seems to very little doubt that the curve $p_1\ (p_2)$ is non-increasing and it is known rigorously that $p_1(0) < 1$.
2. If the expected length of the set of cells on the front F_i at time i grows linearly, the domain is said to *facet*. The threshold for this is of course at least $p^e_.$. Simulations seem to indicate that the two thresholds are equal. As a consequence of faceting the limiting $(p \uparrow 1)$ shapes in the two growth models are simple: square in the first case and a hexagon in the second case with sides parallet to the coordinate axes in both cases.
3. We also note that in Richardson's growth model, which is known to facet [1], the extreme growth can be formulated as above via the Domany and Kinzel model. However it falls to the "easy" regime $p_1 \leq p_2$ ($p_1 = p$ and $p_2 = p$ or $p(2-p)$ depending on how the model is defined.

Numerical bounds for the critical probabilities can be obtained at diferent levels of rigor. Loose but rigorous lower bounds follow from the fact that both of the intersection points in Theorem 2.1 are above the intersection $p_1(p_2)$ with $p_1 = p_2$. This is the critical probability for directed site percolation on a square lattice. Using the best estimate available for it [5] implies that

$$0.70548 \leq p_s^e, \qquad 0.83992 \leq p_t^e.$$

By using the numerical results of [2,7] these bounds can be improved with high degree of confidence to

$$0.73 \leq p_s^e \leq 0.75 \quad \text{and} \quad 0.886 \leq p_t^e \leq 0.912.$$

The last inqualities for the triangular lattice utilize the monotonicity of $p_1\ (p_2)$, bounds for its intersection with the line $p_1 = 1 - p_2$ and the estimate $p_1(0) \leq 0.83$.

2.3. Third phase

By the previous analysis it is clear that when p is sufficiently large there is a third phase which invariant and in fact dominates the homogeneous phases. By domination we refer to the motion of the phase boundary. In this section we try to analyze this phase and find apart from two special cases it is indeed non-trivial.

Again there is a gap between what is evident in the simulations and what can be proved. From simulations it is plain that there is what seems a unique translation invariant disordered phase for all $p > p^g_.$. If it is indeed unique then by statesymmetry it must have density of 1's equaling $\frac{1}{2}$. However all we know rigorously is the follwoing

Theorem 2.2. The $\frac{3}{4}$-voter rule on the square lattice and the 1-voter-rule on the triangular lattice preserve the uniform Bernoulli (product) measure $B(\frac{1}{2})$. None of the other p-voter rules preserve this measure.

Proof. Suppose that we have a $B(\frac{1}{2})$-distribution in a 2×3 cell arrangement on the square latice. Assume that one of the updated cells is 0. By computing the different cases one finds that the p-voter rule gives the conditional probability

$$\Pr(0 \mid 0) = \tfrac{1}{32}(\tfrac{41}{2} - 12p + 8p^2) \tag{2.4}$$

to the other cell update. But then $\Pr(0|0) > \frac{1}{2}$ except at $p = \frac{3}{4}$ where it equals to $\frac{1}{2}$.

For the triangular lattice we consider five cells in a $\nabla\nabla$-arrangement and arrive at the expression $\Pr(0|0) = \frac{1}{16}(10 - 4p + 2p^2)$ which exceeds $\frac{1}{2}$ except at $p = 1$.

The preservation of the uniform Benoulli measure in the triangular case is an immediate consequence of the fact that the 1-voter rule is a permutive cellular automaton (see [4] for permutivity) hence an onto endomorphism of the full shift.

In the square lattice case we need to show that given a finite index set I and the configuration on it $\{a_{(i,j)}\}_{(i,j)\in I}$ the cylinderset $X_I = \{x | x_{(i,j)} = a_{(i,j)}, (i,j) \in I\}$ is of the measure $2^{-|I|}$. For this it suffices to show that given the 1-cylinder $X_{\{(0,0)\}}$ and an $n \times n$ square I containing the origin we have

$$\Pr(X_{\{0,0)\}} \mid X_{I\setminus\{(0,0)\}}) = \tfrac{1}{2}. \qquad (*)$$

To see this we enumerate the columns of I from left to right by I_i, $i = 1, \ldots, n$, and the columns of the $(n+1) \times (n+1)$ base of I in the dual lattice by C_i, $i = 1, \ldots, n+1$. Suppose the origin is not in the rightmost column of I. Then by Lemma 2.3 below, the column C_n is independent of I_n and we have

$$\Pr(X_{\{(0,0)\}} | X_{I\setminus\{(0,0)\}}) = \Pr(X_{\{(0,0)\}} | X_{I\setminus\{\{(0,0)\}\cup I_n\}}).$$

This argument iterated $n-1$ times leads to

$$\Pr(X_{\{(0,0)\}} | X_{I\setminus\{(0,0)\}}) = \Pr(X_{\{(0,0)\}} | X_{I_i\setminus\{(0,0)\}}),$$

where i is the column where origin is located. But along this column we argue with the first part of the lemma and shorten the column by one at a time. Therefore the event $X_{\{(0,0)\}}$ is independent of the rest of X_I and $(*)$ holds. $\square$

Lemma 2.3. Let $N_1 = \{X_{\mathrm{sw}}, X_{\mathrm{nw}}, X_{\mathrm{ne}}, X_{\mathrm{se}}\}$ be a $B(\frac{1}{2})$-distributed configuration in a single 2×2-neighborhood in the square lattice. Suppose X_1 is the update from it under the $\frac{3}{4}$-voter rule. Then any one or two of the variables in N_1 are independent of X_1. More generally suppose we are given an $(n+1) \times 2$ neighborhood of $B(\frac{1}{2})$-distributed cells and the $n \times 1$-update X_n on top of it. Then each of the $(n+1) \times 1$-columns is independent of X_n.

Proof. In the simplest case we add the probabilities of arriving to an update 0 from a neighborhood with at least one 0 and thereby get

$$\begin{aligned}&\Pr(X_{\mathrm{sw}} = 0 | X_1 = 0)\\ &\quad = 2\Pr(X_{\mathrm{sw}} = 0, X_1 = 0)\\ &\quad = \tfrac{1}{8}(\tfrac{11}{2} - 2p)|_{p=3/4} = \tfrac{1}{2}.\end{aligned}$$

By the isotropy (on the lattice) and the 0–1-symmetry of the states this implies that the other conditionings of a single cell have the same value. Hence the first claimed independence follows. In the case of a double from N_1 we have two cases to check depending on whether the cells have a common edge but the essence of the argument in both cases is as above.

For the last part suppose that the statement is true for an update column of height k, X_k, on top of a $(k+1) \times 2$-neighborhood N_k. Let the left column of the neighborhood be C_k. The given argument shows that the case $k = 1$ is true. Let us augment the vector X_k by one cell to a column X_{k+1} and the neighborhood correspondingly to N_{k+1}. Then by the argument above $C_{k+1}\setminus C_k$ is $B(\frac{1}{2})$, independent of its neighbor in C_k and the new cell $X_{k+1}\setminus X_k$. Therefore the entire column C_{k+1} is $B(\frac{1}{2})$-distributed and independent of the column X_{k+1}. $\square$

Remark. The independence properties recorded above only appear at the p-value $\frac{3}{4}$. Moreover even for this rule they do not hold for triples (or quadruples) of cells in N_1.

The result combined with the faceting thresholds indicates an intriguing lattice-dependent difference in the models. In the square case the dominant phase is "ideal", i.e. Bernoulli extremely close to the faceting threshold. Indeed it is tempting to conjecture that the threshold is exactly $\frac{3}{4}$ – this value certainly is within the errorbounds of the numerical studies. By (2.4) on both sides of this p-value the equilibrium seems to support larger contiguous blocks of the same phase. In the triangular case the faceting threshold is well below the Bernoulli case (but the same blocking tendency is evident from Pr(0|0)).

Finally we note that there is a simple heuristic argument that gives the threshold value above. Consider the rightmost cell in a half-infinite vertical facet.

Furthermore suppose that the facet is perfect, i.e. contains all the cells at the level y to the left of the corner cell. In one iterate under p-voter rule the corner moves with probability $p\,\frac{1}{2}$ to the right and with $1-p$ to the left and (and $\frac{1}{2}$ step forward in both cases). Given a left jumps its expected length is $\frac{3}{2}$ hence the corner has a horizontal drift $\frac{1}{2}(4p-3)$. So given a large finite flat boundary piece it should grow linearly for $p > \frac{3}{4}$ whereas for $p \le \frac{3}{4}$ it should vanish almost surely.

2.4. *Deterministic cellular automata*

As mentioned earlier the models considered here are in part motivated by purely deterministic cellular automata. Indeed the reader has already seen them in action – Fig. 2 was generated with such an automaton. We briefly indicate their principle here. Further details can be found in [4].

The random mechanism assumed in Definition 1.1 can be imitated by introducing "hidden states". At every lattice site the symbol 0 or 1 is accompanied by another symbol, call it a digit and denote by d, which for simplicity assumes just two values 0 and 1. The evolution of digits is given by a permutive cellular automaton and is independent of the voter rule. A convenient choice for such automaton is local rule $d_{\text{update}} = \sum d_i \pmod 2$, where d_i's in the appropriate neighborhood (square or triangular) are counted as in Figs. 1(c), (e) and (f).

A permutive cellular automaton is surjective and hence preserves the uniform Bernoulli measure. So if the digits have initially $B(\frac{1}{2})$-distribution then at any given iterate they are independent of each other. This enables pseudo-random dynamics. We can for example tell the voter rule to let the minority win in each 4-1-neighborhood if and only if $\sum d_i = 1$. This event clearly has probability $\frac{1}{4}$ so the cellular automaton imitates a $\frac{1}{4}$-voter rule in this type of neighborhoods. With other choices for the sum we can implement a discrete range of p-values from 0 to 1. In evenly split neighborhoods we can set $\sum d_i \equiv 0 \pmod 2$ to result in 0 winning the update. This yields the desired even probabilities for the two symbols in even neighborhoods. Note that the decision based on $\sum d_i$ has the advantage of being isotropic on digits.

The deterministic cellular automata built using these principles do exhibit behavior extremely close to that of the probabilistic models we have investigated. In particular the critical behavior seems to be the same. This correspondence is similar to the one observed in a more general class of models in [4]. There the dependencies introduced in the construction are further studied.

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