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$\overline{d}$-convergence via a novel block argument [1]

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We relax the uniform Doeblin condition in the $\overline{d}$-convergence theorem in [E1] into a milder condition implied by e.g. the well known Doeblin condition. In doing this we also present a general $\overline{d}$-convergence theorem for processes with jump distributions being arbitrary mixtures of absolutely continuous and discrete type and treat the cases of compact and noncompact statespace in a unified way. This enhances the applicability of the theorem in extending the invariance principles of the theory of Brownian motion.

Introduction

To analyze the stability properties of various stochastic as well as deterministic chaotic systems a novel approach was presented by Ornstein and Weiss in [OW] and [O2]. A new type of infinite-time stability called α-congruence was defined and shown to have most of the characteristics of structural stability. Moreover it accomodates a wider class of systems than just smooth flows and has a number of features of physical importance that the latter lacks. Intuitively one might describe the notion as follows: two dynamical systems are α- congruent if they are measure theoretically isomorphic and the isomorphism moves all but α of the statespace (in the sense of measure) by less than α.

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Stability in the sense of α-congruence has been shown to a number of systems (see [**OW**], [**E1**] and [**E2**]). When the perturbed/approximated Bernoulli process is of infinite entropy the key ingredient in the stability analysis is the calculation of the $\overline{d}$-distance between the processes. In discrete time this metric is the infinite-time generalization of the Hamming distance. In this paper we generalize this argument by relaxing the uniform Doeblin condition of [**E1**] into a more explicit condition (implied by the usual Doeblin condidition) that process. The approximating process is also allowed to have a general jump distribution and the requirement for compactness of the space is removed. This result is in turn easily converted into an α-congruence statement about the indistinguishability of two systems.

1. The $\overline{d}$-convergence

Let (M, d) be a Polish space. Let $\{(X_t^n, P^n)\}_{n\geq 1}$ be a sequence of stationary stochastic processes on (M, d) with stationary distributions $\{\lambda^n\}$ in $\mathcal{P}(M)$. We assume that for P^n-a.e. ω the paths of the process are right continuous and have left limits as $t \in \mathrm{R}_+$ i.e. belong to D. This function space can be thought to be equipped with either the uniform or the Skorokhod topology. In our notation $P_x^n(T)$ denotes the one-dimensional marginal and $P_x^n([0,T))$ the measure on paths on $D([0,T), M)$ that started at $X_0 = x$. If the initial values are unspecified they are assumed to be drawn from the stationary distribution.

By (X_t, P) we denote a non-degenerate diffusion process on (M, d) that has the stationary distribution $\lambda \in \mathcal{P}(M)$. Hence by tightness the process "essentially" lives on a compact set. By the *C-convergence* we mean that for any compact $C \subset M$ we have

$$\sup_{x\in C} \|P_x(t) - \lambda\|_{TV} = o(t).$$

Subsequently the norm is always that of the total variation. (X_t, P) is assumed to be C-convergent.

If M is compact the convergence above can be verified by checking e.g. the Doeblin condition. The rate turns out then to be geometric the exponent being the largest non-zero eigenvalue of the generator of X (see e.g. [**F**]).

Let

$$d_T(X^n, X) = \frac{1}{T}\int_0^T d(X_t^n, X_t)dt$$

and define the following analog of the Prohorov metric:

$$\overline{d}_T(P_{x_1}^n, P_{x_2}) = \inf_{\tilde{\mu}} \inf\{\epsilon > 0 |\ \tilde{\mu}(\{\omega |\ d_T(X^n, X) \geq \epsilon\}) \leq \epsilon\}.$$

Here $\tilde{\mu}$ is a coupling measure with marginals $P_{x_1}^n([0,T))$ and $P_{x_2}([0,T))$. Finally let

$$\overline{d}(P_{x_1}^n, P_{x_2}) = \sup_{T\geq 0} \overline{d}_T(P_{x_1}^n, P_{x_2}) = \lim_{T\to\infty} \overline{d}_T(P_{x_1}^n, P_{x_2}).$$

The last equality is shown e.g. in **[O1]**.

The main result can now be stated.

Theorem: *Suppose* $\{(X_t^n, P^n)\}_{n\geq 1}$ *and* (X_t, P) *are defined as above and that* $P_{x^n}^n \Rightarrow P_x$ *as* $x^n \to x$. *Then* $\overline{d}(P^n, P) \to 0$.

Remark: The weak convergence is necessary but not sufficient condition (as shown in **[E1]**).

Proof: Step 0: Choose a compact set C such that $\lambda(C) > 1 - \epsilon$. If M is compact let $C = M$.

Step 1: By using the C-convergence we obtain T_u such that

$$\sup_{x\in C} \|P_x(T_u) - \lambda\| < \epsilon.$$

Hence

$$\sup_{x,y\in C} \|P_x(T_u) - P_y(T_u)\| < 2\epsilon.$$

Then choose a bounded set C', $C \subset C' \subset M$ such that

$$\inf_{x\in C} P_x(X_t \in C'\ \forall t \in [0, T_u)) > 1 - \epsilon.$$

Let $D = diam(C')$. Choose a coupling time T_c so that $DT_u/(T_u + T_c) < \epsilon$.

Step 2: By the invariance principle $P_{x^n}^n([0,T_c)) \Rightarrow P_x([0,T_c))$ if $n \to \infty$ and $x^n \to x$. Let ρ_{T_c} be the Prohorov metric on the space of measures on $D([0,T_c), M)$.

The function space is separable hence $\rho_{T_c}(P^n_{x^n}, P_x) \to 0$. By using Egorov's Theorem twice we get that for all $\delta \leq \delta_0$ and $n \geq n_0$ it holds that $\rho_{T_c}(P^n_{x^n}, P_x) < \epsilon$ if $d(x^n, x) < \delta_0$ and $x \notin F_1$, $\lambda(F_1) < \epsilon$. Let ${}_{\mathbf{x}}\eta^n_{[0,T_c)}$ denote the corresponding coupling measure. If $\mathbf{x} = (x^n, x) \in M \times F_1$ let the coupling be independent.

Step 3: Let $\mathcal{P}$ be a N atom partition of $C \setminus F_1$ such that P_i's are λ-continuity sets of positive measure and diameter less than $\delta_0/2$. Let $P_0 = (C \setminus F_1)^c$. Also define the following pseudonorms:

$$\|m_1 - m_2\|_{\mathcal{P}} = \sum_{i=1}^{N} |m_1(P_i) - m_2(P_i)|.$$

Clearly $\|m_1 - m_2\|_{\mathcal{P}} \leq \|m_1 - m_2\|_{\mathcal{P}'} \leq \|m_1 - m_2\|$ if $\mathcal{P} \subset \mathcal{P}' \subset \mathcal{B}$.

Step 4: From the invariance principle for $P^n_{x^n}$ and P_x we can in particular deduce that $P^n_{x^n}(T_u) \Rightarrow P_x(T_u)$. Hence for some n_1 we have $\|P^n_{x^n}(T_u) - P_x(T_u)\|_{\mathcal{P}} < \epsilon$ as $n \geq n_1$ for all $x \notin F_2$, $\lambda(F_2) < \epsilon$. Combination of this with Step 1 and the ordering of the norms as above yields

$$\sup_{\substack{x_1 \in C \\ x_2 \in C \setminus F_2}} \|P^n_{x_1}(T_u) - P_{x_2}(T_u)\|_{\mathcal{P}} < \epsilon.$$

Define for $x_1 \in C$, $x_2 \in C \setminus F_2$

$$\begin{aligned} &\rho_{\mathcal{P}}\left(P^n_{x_1}([0, T_u)), P_{x_2}([0, T_u))\right) \\ &\qquad = \inf_{\nu} \inf\{\Delta > 0 |\ \nu(\{\omega |\ X^n_{T_u} \text{ and } X_{T_u} \text{ are in different} \\ &\qquad\qquad\qquad P_i \text{ or one or both are in } P_0\}) < \Delta\} \end{aligned}$$

and call the optimal coupling ${}_{\mathbf{x}}\nu^n_{T_u}$. Since the total variation bounds the coupling error we have the supremum of $\rho_{\mathcal{P}}$ over the given set to be less than ϵ. Off $C \times \{C \setminus F_2\}$ the coupling is independent.

Step 5: Define a (Markovian) coupling on paths on $[0, T)$, $T = T_u + T_c$ from the constructed measures by

$${}_{\mathbf{x}}\mu^n_{[0,T)} = \int {}_{\mathbf{x}}\nu^n_{T_u}(\mathbf{dz})_{\mathbf{z}}\nu^n_{[0,T_c)}.$$

We next show that this is a good coupling on $[0,T)$ in the sense of average distance between the paths. Now

$$d_T(X^n, X) \leq \frac{T_u}{T} d_{T_u}(X^n, X) + \frac{1}{T_c} \int_{T_u}^{T_u+T_c} d(X_t^n, X_t) dt$$

and we denote the first and second terms of the right hand side by I and II respectively. Clearly

$$I \leq \frac{T_u}{T} \left\{ d_{T_u}(X^n, C) + diam(C) + d_{T_u}(X, C) \right\}.$$

From the rarity of the excursions we get that

$$_{\mathbf{x}}\mu_{[0,T)}^n \left(\frac{T_u}{T} d_{T_u}(X, C) > \frac{T_u}{T} D \right) = P_x(d_{T_u}(X, C) > D) < \epsilon \qquad \forall x \in C$$

and by the invariance principle a similar estimate holds for the perturbation as well. Step 1 bounds the rest of I.

In the case of bounded M the bound for II is immediate. In the general case we first write for $\mathbf{x} \in C \times C \setminus F_2$

$$_{\mathbf{x}}\mu_{[0,T)}^n = \int_{z \in F_1 \cup C^c} + \int_{z \in \cup_1^N P_i} {}_{\mathbf{x}}\nu_{T_u}^n(dz^n, dz)_{\mathbf{z}}\eta_{[0,T_c)}^n.$$

On the set $E(\epsilon) = \{\omega \mid \sup_{t \in [T_u, T)} d(X^n, X) > \epsilon\}$ the first integral is bounded by $c\epsilon$ since the second marginal of $\nu_{T_u}^n$ is absolutely continuous with respect to λ and the set $F_1 \cup C^c$ is small by steps 0 and 2. By the fact that the $\mathcal{P}$- variation is small for the chosen $\mathbf{x}$ we get that over $E(\epsilon)$

$$\begin{aligned}
&\int_{z \in \cup_1^N P_i} {}_{\mathbf{x}}\nu_{T_u}^n(\mathbf{dz})_{\mathbf{z}}\eta_{[0,T_c)}^n \\
&\quad = \sum_{i=1}^N \left\{ \int_{z^n, z \in P_i} + \int_{z^n \notin P_i, z \in P_i} {}_{\mathbf{x}}\nu_{T_u}^n(\mathbf{dz})_{\mathbf{z}}\eta_{[0,T_c)}^n \right\} \\
&\quad \leq \sum_{i=1}^N \left\{ c\epsilon \int_{z^n, z \in P_i} {}_{\mathbf{x}}\nu_{T_u}^n(\mathbf{dz}) + c\epsilon \int_{z^n \notin P_i, z \in P_i} {}_{\mathbf{z}}\eta_{[0,T_c)}^n \right\} \leq c\epsilon.
\end{aligned}$$

Since the average distance can not exceed the supremum of the distance we get that $\overline{d}_T(P_{x_1}^n, P_{x_2}) < c\epsilon$. Furthermore by the choice of C we have $\overline{d}_T(P_{\lambda^n}^n, P_\lambda) < c\epsilon$. Call a corresponding $c\epsilon$-good coupling μ_T^n.

Step 6: By induction we get a family of couplings

$$\mu_{kT}^{n} = \int \mu_{(k-1)T}^{n}(\mathbf{dx})_{\mathbf{x}}\mu_{[0,T)}^{n}.$$

Clearly these are again Markovian. Furthermore denote the limit coupling by μ_{∞}^{n}.

Let us now consider the dynamical system $(D_\infty \times D_\infty, \mathcal{D}, \theta_T^n \times \theta_T, \mu_\infty^n)$. As usual θ_t's are the shifts along paths. By a standard argument the product can be chosen to be ergodic (e.g. **[O1]**). But then by the Ergodic Theorem

$$\begin{aligned}\frac{1}{kT}\int_0^{kT} d(X_t^n, X_t)dt &= \frac{1}{k}\sum_{i=0}^{k-1} d_T((X^n, X)\circ(\theta_{iT}^n, \theta_{iT})) \\ &\to \int_{D_\infty \times D_\infty} d_T(\mathbf{x})\mu_T^n(\mathbf{dx}) \qquad\qquad \mu_\infty^n\text{—a.s..}\end{aligned}$$

By Step 5 the last expression is bounded by $c\epsilon$. Therefore the proof is complete. ∎

2. Applications

Our result is directly applicable to all the cases considered in **[E1]**. But since the requirement for the uniformity in the tail of the random walk sequence is now removed we can in fact expect a wider range of applications. This includes sequences of dependent random walks for which an invariance principle is known (see e.g. **[EK]**) as well as deterministic dynamical systems with only a minimal amount of "seed" randomness in them.

We also note that all the aforementioned convergence results extend to α-congruence when the product of the finite-entropy (approximating) process with an infinite-entropy Bernoulli viewer process is formed. This enables the construction of an isomorphism between two Bernoulli systems. For the Bernoulliness of the billiards and the details of the construction we refer to **[OW]** and **[E2]**.

Bibliography

[E1] Eloranta, K.V.: α-congruence for Markov Processes, to appear in the Annals of Probability, 1990

[E2] Eloranta, K.V.: α-congruence for Dispersive Billiards, preprint, 1989

[EK] Ethier, S.N., Kurtz, T.G.: *Markov Processes: Characterization and Convergence*, J. Wiley, 1986

[F] Freidlin, M.: *Functional Integration and Partial Differential Equations*, Princeton, 1985

[O1] Ornstein, D.S.: *Ergodic Theory, Randomness and Dynamical Systems*, Yale, 1974

[O2] Ornstein, D.S.: Ergodic Theory, Randomness, and "Chaos", *Science* **243** (1989), pp. 182-186

[OW] Ornstein, D.S., Weiss, B.: Statistical Properties of Chaotic Systems, preprint, 1989

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