



The bounded eight-vertex model[☆]

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Abstract

The bounded version of the eight-vertex model of Statistical Mechanics is investigated. We study square, diamond and general finite domains on the square lattice and give exact characterizations to legal boundary conditions and number of fill-ins. The sets of legal configurations with a given boundary turn out always to have the graph topology of a hypercube with a particularly simple edge action. This enables a simple probabilistic description of the configurations as well as an efficient configuration generation using a cellular automaton. Finally, by invoking height functions we study restricted edge action which leads to ice-model as well as to lesser known vertex models, some subsets of the eight-vertex model, some not.

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0. Introduction

The attempts in recent years to extend the one-dimensional theory of symbolic dynamics to higher dimensions have uncovered both challenging problems and yielded surprising successes. On one hand, there are no-go results rooted to undecidability and on the other hand completely new phenomena that manifest themselves only trivially in one dimension. The former stem from the theory of tilings and in particular the fundamental work of Berger (see [7]), whereas the latter are closely related to classical Statistical Mechanics formulations.

In this paper we consider a well-known Statistical Mechanics model, the planar eight-vertex model. The infinite model of immediate physical interest has been studied

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earlier in great detail [1]. Also the toral case has received attention but as it has no boundary, none of the geometric subtleties associated with the boundary dependency will show up there. Here we study this bounded case in finite domains hoping that the results on boundary dependency can help to clarify the phenomena of long distance order in this and the embedded models like the ice-model (which was analyzed in the companion work [6]).

While our model is of physical nature, the results can be viewed as a part of a bigger program that attempts to bring unity to the theories of symbolic dynamics and tilings. The original impetus to this work came mainly from a group theoretic study of polyominoes [3,11] which was later extended by others, notably in [8]. These studies concentrated on the tileability of a finite planar region with given primitives, in these cases dominoes or polyominoes.

It turns out that many Statistical Mechanics models can be treated in similar fashion. Instead of tiling with polyominoes we can, for example, distribute arrows between neighboring lattice sites according to a fixed set of local matching rules at vertices. Models of this type include the ice-model, several color-models, the eight-vertex model and still others, some of which appear for the first time in this paper. Through simple coding these turn into tiling problems and again the shape of the domain can play a critical role in determining the generic properties of the tilings.

Specifically, we consider the tileability and counting problems for the eight-vertex model and indicate a simple but rather general way of generating all the allowed configurations with cellular automaton. This is a consequence of a connectivity result that seems to underlie many different Statistical Mechanics models.

Our analysis also shows that independent of the domain shape the space of finite eight-vertex configurations has the particularly pleasing topology of a hypercube. The final chapter analyzes the model using the concept of height. We give a natural explanation why a non-trivial height function cannot exist for the eight-vertex model, but it does exist for a number of interesting subsets thereby giving rise to ice and other models with critical boundary dependency.

1. Setup and size

In this section we first define the model and then analyze it on two different types of finite domains. This involves characterizing legal boundaries, solving the fill-in problem and computing the size of the set of legal configurations.

Consider the square lattice in two dimensions, $\mathbb{Z}^2$. Unlike in most statistical mechanics lattice models the vertex models do not have any spin, etc. variables associated to the lattice points. Instead the variables are now the arrows between four nearest-neighbor sites.

Definition 1.1. A vertex configuration at a lattice site in $\mathbb{Z}^2$ is legal for the eight-vertex rule if there are either 0, 2 or 4 incoming arrows and the rest are outgoing. A configuration is legal if it has an allowed vertex configuration at every lattice site.

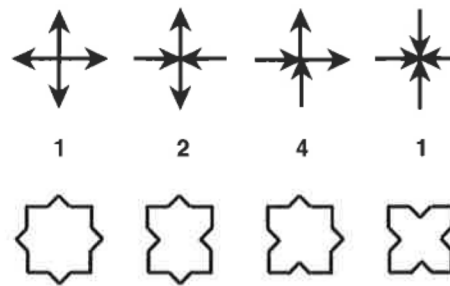


Fig. 1. Vertex configurations and prototiles with multiplicities.

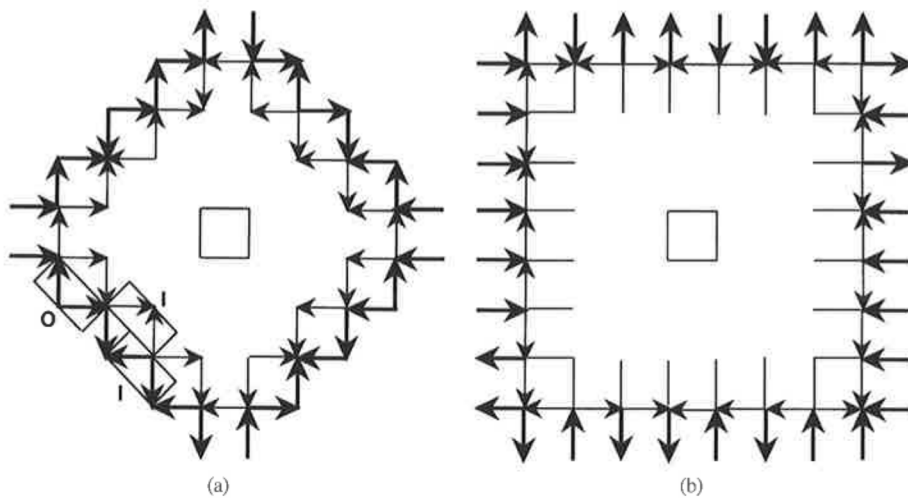


Fig. 2. (a) Diamond and (b) square domains.

The allowed vertex configurations are illustrated in Fig. 1. The numbers indicate the multiplicity of the arrangement. There are eight possibilities, hence the name of the model. Alternatively, the bumps and dents in the prototiles encode the eight-vertex rule thereby converting a legal configuration into a tiling.

The model on the infinite $\mathbb{Z}^2$ -lattice as well as on a finite torus has been studied before (e.g. [1]). Both of these cases are boundaryless. In order to study the boundary dependency we need to define a suitable finite domain and the arrow configuration on the boundary.

The domains that we will consider first are the diamond and square which differ in the orientation with respect to the lattice axes. The boundary arrows to be specified are obviously somewhat different. We will first derive the counting result for the diamond since it has the cleanest boundary condition of all domains.

N -diamond is a subset of $\mathbb{Z}^2$ of height/width of N lattice sites. The arrow configuration on it has $N - 2$ arrows along each of the four diagonal sides. Fig. 2,

left, illustrates a 10-diamond. We assume for simplicity that $N \geq 2$ is even. The total number of arrows in a N -diamond is $N^2 - 4$ and the number of lattice sites $N^2/2 + N$. We omit the corner arrows as they are superfluous for the purpose of fill-in.

The *boundary configuration* of the N -diamond, which consists of $4N - 8$ arrows along the sides, is fixed. It can in principle be chosen arbitrarily but our first problem is to solve when a given boundary configuration can be extended to a complete configuration of the interior. To this end it is useful to partition the configuration into *shells* as indicated in Fig. 2 (the boundary is distinguished by bold arrows and the next smaller shell by light arrows).

On a shell we distinguish two types of arrow pairs. If two neighboring arrows on the shell point to or away from the common lattice point we say that they form a *switch block* and call the lattice point a *switch point*. If one of the two neighboring arrows on a shell points in and one out of the common lattice point we say that they form a *neutral block*. Furthermore, if the switch point is on the inside of the shell we call the block an *inside switch block* and outside otherwise. These are marked with “I” and “O” in the figure.

With these definitions we are ready to formulate a few basic observations:

- (1) By the eight-vertex rule the existence of an inner switch block on a shell implies the existence of an outer switch block at the next smaller shell. The switch blocks share a common switch point. However, the inner switch block does not force the type of the outer switch block: it can be both arrows in or both out.
- (2) The total number of switch points on a shell must be even. Here we record switch points in both inner and outer switch blocks. This is just a parity count—when we traverse the shell the direction of the arrows changes at every time we cross a switch point. Hence, when we arrive back to the initial arrow we must have seen an even number of switch points.
- (3) (1) and (2) immediately imply that if the boundary has an even (odd) number of inner switch points, then all the inner shells must have an even (odd resp.) number of switch points, inner and outer.
- (4) The smallest shell (little square in the figure) can be filled in iff the next larger shell has an even number of inner switch blocks.

The *flux* across a loop around a lattice point is either 0 or ± 4 . Loop here is a clockwise-oriented unit square in the dual lattice $(\mathbb{Z} + \frac{1}{2})^2$ centered at the lattice point. A loop around a set of lattice points is a sum of such unit loops hence the flux across it has to be divisible by four.

Consider now the set of lattice points in the N -diamond with four arrows attached (i.e. omit the extremal lattice sites in the N -diamond). Define the boundary flux, F , to be the flux across the loop around this set. Facts (1)–(4) imply that the fill-in shell by shell from the boundary is successful iff on the boundary there is an even number of inner switch points. Every switch point contributes ± 2 to the boundary flux. Neutral blocks contribute 0. Hence, the fill-in is possible iff F is divisible by four—the same conclusion as in the previous paragraph. This flux condition equals to the requirement that there is an even number of arrows pointing in. Call a boundary arrow arrangement that has such property a *legal boundary*.

The boundary minus the four corner arrows is determined by its partition into inner switch or neutral blocks. There are $2N - 4$ of them in an N -diamond. Using an elementary binomial identity we find that the total number of ways that the boundary blocks can be chosen legally is

$$\sum_{\substack{k=0 \\ k \text{ even}}}^{2N-4} \binom{2N-4}{k} 2^k 2^{2N-4-k} = 2^{2N-5} \sum_{k=0}^{2N-4} \binom{2N-4}{k} = 2^{4N-9}.$$

Let us now examine the fill-in choices. For that purpose we number the shells from outside in such a way that the boundary is the first shell, the next largest is the second and so on. By the preceding argument the i th shell partitions into $2N - 4i$ inner blocks, switch or neutral.

The key fact that enables the counting is

- (5) The locations of the inner switch blocks on a shell can be chosen independently of the locations of inner switch blocks on other shells. Or equivalently the location of inner switch blocks on a shell is independent of the location of outer switch blocks on the shell.

The equivalence follows immediately from Fact 1 above. Note that the statement does only refer to location and not to type.

Given the shell $i - 1$, the counting argument above slightly refined gives that the i th shell can be chosen in

$$2 \sum_{\substack{k=0 \\ k \text{ even}}}^{2N-4i} \binom{2N-4i}{k} = 2^{2N-4i}$$

different ways. Factor 2 in front is due to the fact that besides the switch point locations we can choose the direction of exactly one arrow on the shell.

By (5) the total number is then obtained by multiplying the shell contributions

$$2 \prod_{i=2}^{(N/2)-1} 2^{2N-4i} = 2^{N^2/2-3N+5},$$

where the 2 in front comes from choosing one arrow direction in the smallest shell (the only choice there).

We can summarize the above as the first existence and counting result.

Theorem 1.2. *An arrow configuration on a diamond boundary can be extended to an arrow configuration on the entire set iff $F \equiv 0 \pmod{4}$. There are 2^{4N-9} such legal boundaries for an N -diamond. Each of these extends in $2^{(N^2/2)-3N+5}$ ways to a complete arrow configuration of the interior. The total number of N -diamond configurations is $2^{(N^2/2)+N-4}$.*

We will postpone interpreting this until we have analyzed the square domain case as well.

The N -square is the domain that consists of N^2 lattice points and $2N^2 + 2N$ arrows as indicated in Fig. 2b (a 8-square). The $4N$ arrows that have been rendered bold have to be specified as a boundary condition. For simplicity let N again be even.

It is again useful to distinguish a shell. In Fig. 2b the first shell is the one marked with light arrows. The smallest shell (the $(N/2)$ th, here unoriented) is shown as well. The reason for this shell choice is evident; given the boundary arrows, once we choose the arrows on the neighboring shell a new inner boundary is uniquely determined (the unoriented arrows on the inside of the first shell in the figure) and we can proceed inductively. As in the diamond case, in the square case, the total flux along the arrows on the boundary, has to be divisible by four. Hence, there are total of $\sum_{k=0, k \text{ even}}^{4N} \binom{4N}{k} = 2^{4N-1}$ legal boundary conditions.

Compatible with Fact 2 in the diamond context we must record an even number of arrow direction reversals on the shell as we traverse it once. Call the lattice points where this happens again switch points. The location and number of corner switch points we cannot choose as they are determined by the next larger shell. But others on the shell we can among the $4(N-2i)$ possible locations on the i th shell. Depending on whether there is an even or odd number of corner switch points we have to pick even or odd number of off-corner switch points on each shell. But in either case there are the total of $2^{4(N-2i)}$, $1 \leq i \leq N/2 - 1$ choices. Here we have also accounted the choice of one arrow orientation after which the shell is completely determined. For $i = N/2$ (the center shell) there are two choices as in the diamond case.

The shells were chosen the given way to have the independence of the choices as in the diamond case. Now the locations (hence also the count) of the off-corner switch points on neighboring shells are independent. Therefore, we can compute the totality of choices as

$$2 \prod_{i=1}^{(N/2)-1} 2^{4(N-2i)} = 2^{(N-1)^2}.$$

Theorem 1.3. *An arrow configuration on a square boundary can be extended to an arrow configuration on the entire set iff $F \equiv 0 \pmod{4}$. There are 2^{4N-1} such legal boundaries for an N -square. Each of these extends in 2^{N^2-2N+1} ways to a complete arrow configuration of the interior. The total number of N -square configurations is $2^{N(N+2)}$.*

Remarks. (1) Although the geometry of the domain forces a somewhat different argument in the two cases it does not alter the number of choices in a significant way. One notes that the square domain has approximately twice as many lattice points and arrows but essentially the same number of boundary arrows. In particular, the asymptotics like topological entropy agree. This quantity for a vertex model is the maximal “uncertainty per arrow”. More formally it is

$$h_{top} = \lim_{M \rightarrow \infty} \frac{1}{M} \log(\{\text{total number of } M - \text{arrow configurations}\}),$$

where the domain of size M will retain its shape as its size increases (to avoid essentially one-dimensional limiting shape). The theorems imply immediately the lower bound $\frac{1}{2} \log 2$ for the topological entropy of the infinite model. In fact, the bound is the exact value since we are imposing no boundary condition in the last statements of the theorems. This number is approximately 0.346574. For comparisons sake we mention that for the infinite free model $h_{top} = \log 2 \approx 0.69315$ and for the (more restrictive) six-vertex model $h_{top} = \frac{3}{4} \log \frac{4}{3} \approx 0.215761$.

(2) These results indicates a striking homogeneity in the model: all legal boundary conditions in the given geometry have equal number of fill-ins. It reminds of the situation to the one encountered in finite groups, the fill-ins corresponding to the cosets of a group. Later we will see what the action generating each *coset* is.

(3) The results extend immediately to a rectangle standing on its corner and a lattice rectangle.

2. Irreducibility

In this section we investigate the “perturbations” of the allowed configurations. This yields a simple characterization of the topological structure of the set of configurations. From it we obtain a constructive method to generate the configurations and to analyze their probabilistic properties.

The first observation is that for each of the eight-vertex configurations we can simultaneously reverse the directions of two arrows and the vertex configuration remains legal. This flip at just a single-vertex configuration violates the rule at two of its nearest neighbors. But if we reverse the arrows along any closed arrow loop—thereby forming a *disagreement loop*—in the resulting configuration all vertex configurations are again legal. Note that while this loop/path consists of arrows it does not need to be directed as a whole.

Definition 2.1. A 1-loop is the quadruple consisting of arrows connecting four neighboring lattice sites in square formation. The reversal of all the arrows in such a loop is an elementary move.

We immediately note the useful property that elementary moves commute. Subsequently, the symbol for an elementary move refers to the coordinates of the dual lattice site where it is performed.

Reversal of 1-loops connects some set of configurations. The natural question then is to characterize this set, i.e. the configurations that can be constructed from a given configuration using a finite sequence of elementary moves. Note that in the case of a bounded domain with a fixed boundary, a loop reversal can never produce a configuration in the other cosets as the path to be reversed cannot contain any boundary arrows.

Call the action of 1-loop reversals *irreducible* on a set of configurations, C , if any two elements of C can be transformed to each other with a finite sequence of elementary moves.

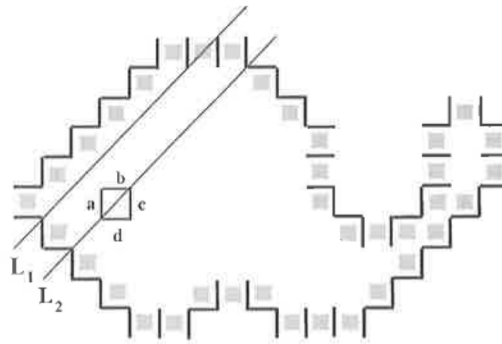


Fig. 3. The sweep.

The domains on which we show the irreducibility are more general than those in Section 1. Call the connected boundary segment on the diamond edge a *staircase* and call a column/row of parallel arrows, as on the square boundary, a *ladder*. The boundary of a *legal domain* consists of arbitrary finite segments of staircases and ladders. We assume that they are enclosing a finite set of lattice point which is simply connected (no off-boundary loop encloses boundary arrows).

Let D_{free} denote the union of unit squares in the domain with arrows on its edges, none of which is a boundary arrow. $A(D_{free})$, the area of D_{free} , is a finite integer, the area of the interior of the domain. Fig. 3 illustrates a legal domain. The small gray squares denote those unit squares in the domain which are not free.

Theorem 2.2. *Elementary moves are irreducible on any set of configurations on a legal domain and with a common boundary configuration. They connect the $2^{A(D_{free})}$ distinct configurations on the domain.*

Proof. Consider two legal arrow covers of the domain D with same arrow configuration on the boundary. Call these configurations $\mathbf{x}$ and $\mathbf{y}$.

For simplicity we first identify all the disagreements on the two configurations. It is easy to see that they are all on disagreement loops. The loops are obviously off-boundary and they can be chosen to be disjoint (possibly sharing a vertex but not an edge). The loops can be nested but we ignore all except the maximal ones which enclose the smaller loops inside them. Note that there are no boundary arrows on the maximal disagreement loops or in their interior because of the simply connectedness. So all 1-loops in this maximal disagreement set can be reversed.

Call the sections of the configuration-oriented SW-NE diagonals. We compare the two configurations lexicographically, diagonal by diagonal, and change $\mathbf{y}$ locally if needed, and at the end of the comparison $\mathbf{x}$ and the image of $\mathbf{y}$ will be identical.

Starting on line L_1 (see Fig. 3) at the NW extremity of the domain we record from the left to the right the agreements on the 1-loops centered on it if they are within the maximal disagreement set. To be precise we only check the arrows on the W-edge of each of the 1-loops (in $\mathbf{x}$ and $\mathbf{y}$) at the same dual lattice point on the line. If no

disagreement is found the arrows on both W- and N-edges of the 1-loops agree in the two configurations and we parallel transport the line to its next location, one unit down.

Suppose there is a first disagreement in a 1-loop on L_2 at the site illustrated in the figure. Then in fact both arrows a and b disagree. This is because the two arrows in the same vertex configuration on the previous diagonal have already been found to agree on the two configurations. Now apply an elementary move to the quadruple (a, b, c, d) in the configuration y . After this the N and W arrows agree at this 1-loop in both configurations. We continue to the next 1-loop on the diagonal as long as it is within the maximal disagreement set. Clearly, the process can be continued to the end of the last diagonal intersecting the maximal disagreement set, after which x and the image of y agree.

Since one sweep across the configuration suffices to connect, the argument above implies that the domain D with the given boundary configuration has at most $2^{A(D_{free})}$ distinct arrow covers compatible with the eight-vertex rule. Suppose there are less. Then there are distinct sequences of elementary moves $\{p_i\}_{i=1}^n$ and $\{r_i\}_{i=1}^m$ (coordinates of the sites where an elementary move is performed) such that $p_n \cdots p_1 x = r_m \cdots r_1 x$. Since an elementary move is an involution this implies $x = p_{n'} \cdots p_1 r_{m'} \cdots r_1 x$, for some $n' \leq n$, $m' \leq m$, where we have by commutativity canceled all elementary moves performed at the same sites. By assumption $p_{n'} \cdots p_1 r_{m'} \cdots r_1$ cannot be an identity. Hence, the equation fails for all x and we conclude the cardinality in the statement. $\square$

Corollary 2.3. *The action of the elementary moves is irreducible on the set of diamond or square configurations with identical boundary arrows. Maximum number of elementary moves needed for a N -diamond is $N^2/2 - 3N + 5$, $N \geq 4$ and for a N -square it is $(N - 1)^2$, $N \geq 2$.*

Proof. There are restricted 1-loops along the boundaries in the diamond case. Once we remove them there are $(N/2 - 1)^2 + (N/2 - 2)^2 = N^2/2 - 3N + 5$ free 1-loops left in the configuration on a N -diamond. The N -square consists of $(N - 1)^2$ 1-loops, all free. $\square$

Corollary 2.4. *The set of configurations is a hypercube in $A(D_{free})$ dimensions the elementary moves being the coordinate shifts.*

Proof. By the theorem the set of configurations on D with a given boundary is a connected graph with $2^{A(D_{free})}$ nodes. There are no self loops and each node has exactly $A(D_{free})$ nearest neighbor configurations reached with a single elementary move. $\square$

Remarks. (1) The tileability condition $F \equiv 0 \pmod{4}$ holds for general domain as well. However, counting the exact number of legal boundary conditions without specific knowledge of the domain geometry seems complicated.

(2) Note that the graph diameter results in Corollary 2.3 necessarily have to agree with the coset size exponents formulated in Theorems 1.2 and 1.3. The reason why we

present the theorems of Section 1 as well as Theorem 2.2 is that their proofs illustrate the independence embedded in the model in quite different ways. The argument given for Theorem 2.2 is more robust though. It applies to various domain shapes while the shell construction in the proofs of Section 1 becomes rather unwieldy for complicated domain shapes.

(3) This type of connectivity result seems to hold with some generality once the correct elementary moves have been identified. It has been shown to the ice model [6,9], to dominoes in greater generality [10], to lozenge tilings, etc. Sometimes it almost holds, failing in an interesting way for a small subset of “exotic” configurations [5].

The topological findings above immediately imply a simple characterization to a generic configuration. Fix a legal domain D and a legal boundary condition for it. Let the interior, D_{free} , be defined as above. Pick an arrow site which is an edge of one of the unit squares in D_{free} . Partition the set of legal configurations, C , into subsets C_+ and C_- according to whether the arrow at the selected site points up or down (to the right or left if horizontal). Since an elementary move is a 1–1 map and flips the test arrow in a 1-loop in D_{free} containing it, we conclude $|C_+| = |C_-|$. Hence, if the configurations are equally weighted the probability of seeing a particular arrow orientation at any site in the interior of the domain is exactly $\frac{1}{2}$. This homogeneity of the configurations, i.e. lack of long-range dependency from the boundary configuration is distinctly different from the phenomena observed in, e.g. the ice model [6].

The configurations are easy to generate. Denote the set of configurations understood as a $A(D_{free})$ -dimensional hypercube by H . Let X_n be a nearest-neighbor random walk on H with independent jumps and uniform transition probabilities. Hence, the jump to each of the neighboring configurations has the probability $1/A(D_{free})$. The equilibrium distribution is then of course uniform, i.e. each configuration has probability $2^{-A(D_{free})}$. Starting from any legal configuration the walk converges exponentially fast to the uniform distribution on H . The implementation of this walk as a probabilistic cellular automaton is straightforward. For explicit details we refer to the companion paper [6] (and [4] for a computationally efficient *deterministic* scheme that can be applied to the case at hand as well).

3. Height

A height function is an analytic device frequently useful in the context of lattice models. Roughly speaking it is a function defined on the configurations which keeps track of certain regularities in the configurations. The name follows from the interpretation of the graph of this function lying as a surface above the configuration. They have been shown to exist, e.g. for dominoes (for a nice treatment of this context see [11]) and for the ice-model [2,6]. The eight-vertex model does not have a height function but understanding why this is the case leads to further understanding of the homogeneity of the configurations as well as to studying certain interesting subsets of them via the set of elementary moves.

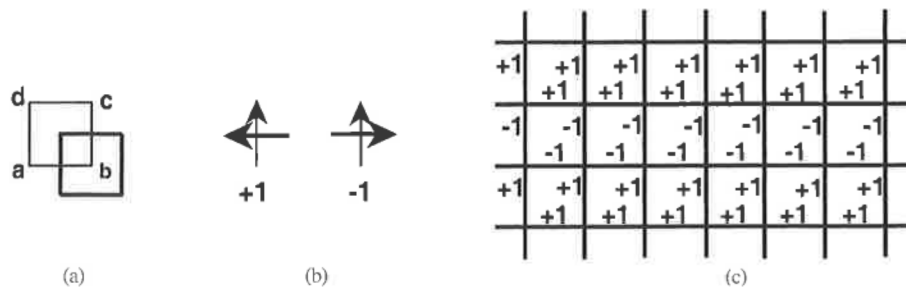


Fig. 4. Height and weight.

Since we will consider briefly also other vertex rules than the one for eight-vertex model, we state the defining properties for height more generally. By vertex model we mean a model on the given lattice where specifying the vertex configurations of arrows at every lattice point defines the configuration uniquely. Denote the set of configurations again by C .

Definition 3.1. Height function h for a vertex model on $\mathbb{Z}^2$ is an integer-valued function on $C \times (\mathbb{Z} + \frac{1}{2})^2$. The value of $h(\mathbf{x}, \cdot)$, $\mathbf{x} \in C$ changes by ± 1 from a dual lattice point to its neighbor, the sign depending on the heading (left/right) of the configuration arrow crossed.

Remarks. (1) The function is uniquely defined up to an additive constant, which is fixed by defining the value of h at one point.

(2) To have a well defined function h , the value that it returns at the end of a closed loop has to agree with the initial value. Or equivalently the value of h at a given dual lattice site is independent of the path along which it is computed from the base point (where the value is known).

Using the results of Section 2 it is now easy to see that the eight-vertex model does not admit a non-trivial height function (a trivial height function computes height only mod 2). Consider the arrangement in Fig. 4a. (lattice lines are bold, dual lattice lines light, $a, \dots, d$ denote dual lattice sites).

Suppose for simplicity that $h(\mathbf{x}, a) = 0$. The height $h(\mathbf{x}, c)$ can be computed, e.g. along the two shortest paths that pass through b and d , respectively.

Consider now the path through b . The height $h(\mathbf{x}, c)$ depends on the orientations of the two arrows along the western and northern edges of the bold square. If we now perform an elementary move on this square these arrows are reversed and the height at c will be $h(\mathbf{x}', c) = -h(\mathbf{x}, c)$ ($\mathbf{x}'$ is the configuration after the move). But along the path via d the arrows are unchanged so necessarily $h(\mathbf{x}, c) = h(\mathbf{x}', c) = 0$.

The only choice in the definition of the height function is on the weights we assign on the arrow orientations (whether an arrow pointing to the right of the height path counts $+1$ or -1). Suppose that the two arrows reversed were attached to each other

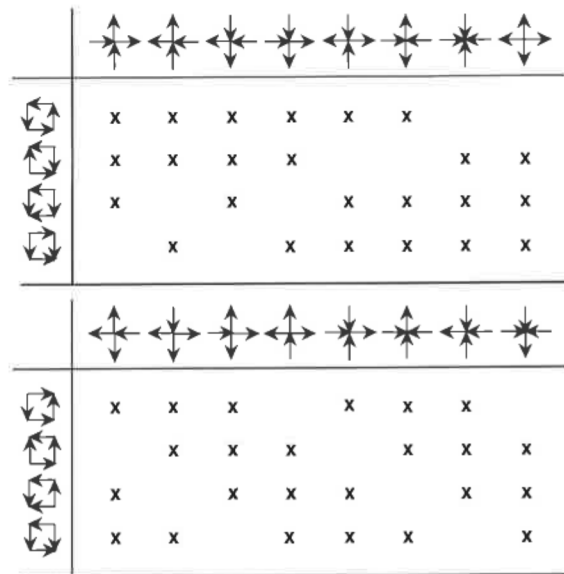


Fig. 5. Embedded vertex rules.

head–tail. Then we must read them for the height with same weight in order to obtain the value 0 at c . But if they were pointing head–head or tail–tail then necessarily $h(\mathbf{x}, c) \neq h(\mathbf{x}', c)$, a contradiction.

The argument above can be loosely summed by saying that the reason why the eight-vertex model does not have a height function is because the set of elementary moves is too big. This in turn suggests some questions: Are there more restricted subsets of this set of moves which generate interesting (say in the sense of height) sets of configurations? Can the eight-vertex model perhaps be generated with a subset of the elementary moves? What kind of height functions are possible? These are the questions that we concentrate on now.

The leftmost column in Fig. 5 decomposes the set of elementary moves as stated in Definition 2.1 into its primitives, reversals of eight distinct arrow loops. The top row indicates all the possible vertex configurations on the square lattice with 0, 2 or 4 incoming arrows and the middle those with 1 or 3 incoming arrows. The entries in the matrix denote the vertex configurations legal in the configurations on which the *primitive move* represented on the left acts. We are not going to analyze every model (i.e. row) in the table but rather give the principles according to which it can be done and the entries in the matrix decided. These are indeed quite simple:

- (1) Choose one of the primitive moves on the left.
- (2) Determine the height function from the primitive move. This is equivalent to deciding how the ± 1 weights are assigned to the arrows in the four different orientations as the height is computed along a dual lattice path. Once this has been accomplished the height is well defined (path independent) for the “perturbed”

configuration if it was for the original one. Everyone of the primitive moves listed admits a height function on the configurations compatible with the moves.

- (3) Using the height single out the legal vertex configurations using the fact that height difference around a closed loop must vanish.

Let us illustrate this procedure in a couple of cases. In order to remove the ambiguity in the choice of the height we choose the sign convention as indicated in Fig. 4b. The lattice arrow is again bold and the dual lattice path along which the height is computed is light. The arrangements and their rotations indicate how the height changes when the arrow crossing is recorded with positive weight.

Example 3.2. Suppose we have chosen the primitive action to be the reversal of *directed* 1-loops as on the top of the column on the left in Fig. 5. Consider now the arrangement as in Fig. 4a where the bold arrows form a directed 1-loop. To have $h(\mathbf{x}', c) = h(\mathbf{x}, c) = h(\mathbf{x}, a) = 0$ we need to record one +1 and one -1 along the height path $a \rightarrow b \rightarrow c$. But since there is indeed one left going and one right going arrow on the path zero will result if we record as indicated in Fig. 4b. Same obviously holds for the three rotations of the arrangement in Fig. 4a. Hence, the height is simply computed from increments as in Fig. 4b, i.e. weights are all +1 (or all -1).

Height increase around a closed loop must be zero. Using the height increments of Fig. 4b around a 1-loop in the dual lattice immediately singles out the vertex configurations marked with crosses in Fig. 5, first row in the matrix.

The reader may recognize this vertex model. It is the classical six-vertex or ice-model of Statistical Mechanics [1]. In [6] it is shown that the given primitive action connects all the configurations (made from the six legal-vertex configurations) exactly in the same sense as elementary moves connect the eight-vertex configurations in Theorem 2.2. (the geometry of the domain is more restricted in [6] though).

Example 3.3. Consider the primitive move on the first row of the bottom half in Fig. 5. Arguing as in the previous example on the path independence of the height in Fig. 4a now gives a different conclusion. The height is consistent with the primitive move if and only if the increments of Fig. 4b (these arrangements and their rotations) are weighted as in Fig. 4c (these weights or all signs reversed). Using this weighted height to test the vertex configurations results in the entries on the top row of the bottom half of the table.

The weights defining the height function are all +1 or all -1 for the directed 1-loop reversal (i.e. ice-model of Example 3.2.) and non-trivial and distinct for the heights of all the other seven primitive moves in Fig. 5. Note that this implies that one cannot define a height function for the eight-vertex model even if we restrict to a subset of the set of elementary moves (recall from the beginning of the section that the reason height did not exist was that the set of elementary moves was too large). For example the two primitive moves in the first two lines of the matrix admit together all the eight-vertex configurations; yet, neither of the two height functions is well defined on all the vertex configurations, i.e. neither extends to the “full” model. Whether a strict

subset of the elementary moves actually generates the eight-vertex configurations on a legal finite domain (in the sense of Theorem 2.2.) is an open problem.

By definition height $h(\mathbf{x}, a)$ is a Lipschitz-function in a for all configurations $\mathbf{x}$. Whenever its discrete partial derivatives in a are constant ± 1 in some neighborhood of a the configuration is ordered there in some fashion. Conversely, one could think of disordered configurations to be those whose average (in a) height increments are near zero.

Study of the ice-model using height indicates that the domain geometry and the boundary configuration can influence the interior of a configuration in a drastic fashion [6]. Constant height derivative on a boundary segment forces the vertex configuration on a wedge in the interior of the domain. The models listed in Fig. 5 all share this property. This is because their configurations can be mapped 1–1 to ice configurations once the height function is known. So in conclusion we note that although the eight-vertex model has neither a height function nor long-range boundary dependency, its configuration set has naturally defined subsets with these properties.

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