# THE INTERACTION DYNAMICS OF THE KINKS IN THE CELLULAR AUTOMATON RULE 18

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We present the results of a series of computer simulations on the joint motion of two kinks in the elementary cellular automaton Rule 18. These show a remarkably close approximation to the case of independent annihilating random walks except at close range interaction where combinatorial conditions impose an asymmetry on the dynamics. Attempts to analyze the underlying state transition graph are reported. The results are central in clarifying the dynamics in the general case of an ensemble of kinks.

**Keywords:** Cellular automaton, annihilating random walk, emergence of order

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## Introduction

In recent years a lot of attention has been focused on interactive dynamics like particle systems and cellular automata ([**D**], [**W**]). In these systems the site variables are usually very simple and so is the interaction rule yet a great variety of dynamics can emerge. In particular striking examples where (pseudo-) randomness is created by purely deterministic means have been found if not fully analysed. The converse, emergence of order out of randomness (e.g. a random initial condition), is understood only in a rather primitive fashion.

In the context of elementary cellular automata Rule 18 variations of both of these intriguing phenomena surface. On one hand the rule transforms the randomness in the initial configuration into perfect diffusivity that prevails at all later times. This was discovered in simulations by Grassberger ([**G**]) and was later analyzed rigorously by e.g. Lind ([**L**]) and Eloranta & Nummelin ([**EN**]). On the other hand more ordered configurations are formed through joining of two adjacent phases. This is facilitated by an annihilation mechanism in which neighbouring kinks disappear pairwise. The mechanism together with the diffusivity is conjectured to determine the asymptotics of the Rule 18.

In this paper the central problem of the joint motion of two kinks is studied. We find it very close to the motion of two independent annihilating random walks except for close range interaction. There certain combinatorial constraints deny the kinks independent-like movement. But even then the statistics are close to those of the independent case. In particular we study the the spatial and temporal statistics of the difference process as well as the correlations of the movements of the kinks. In spite of obvious depency between the kinks the motion is remarkably close to independent and the study finds no reasons to refute Lind's conjectures ([**L**]). It seems likely that the observed division into two interaction regimes could be utilized in proving results about the dynamics of an ensemble of kinks.

## 1. Definitions

This study is a sequel of [**EN**] and we build our definitions and notation on this reference.

Let $S = \{0, 1\}$ be the set of symbols and $E = S^{\mathbf{Z}}$ the set of configurations on the lattice of integers $\mathbf{Z}$. Equipped with the product topology it becomes a compact metric space homeomorphic to the Cantor set. A *one-dimensional elementary cellular automaton* is a dynamical system on $E$ defined by a blockmap on three neighbouring symbols which commutes with the shift on $E$. The blockmap of *Rule 18* is simply $001 \mapsto 1$, $100 \mapsto 1$ while other triples map to zero.

The image of a configuration $\eta = \{\eta(x), x \in \mathbf{Z}\}$ under the Rule 18 is denoted by $\tau\eta$. A partial configuration is denoted by $\eta[a, b] = (\eta(a), \ldots, \eta(b))$, $a \leq b$, $a, b \in \mathbf{Z}$.

A configuration $\eta$ contains a *kink* (*dislocation*) if $\eta[a, b] = (1, 0, \ldots, 0, 1)$ for $b - a$ odd. The midpoint of the kink $(a + b)/2$ belongs to $\mathbf{Z}+1/2$.

We observe that when $\eta[n, n+1] = (1, 1)$ the kink is bound to expand with the next iterate of $\tau$. By saying that the kink *jumps* we refer to the possible movement of its midpoint.

Let $E_0 = \{\eta | \ \eta(2j) = 0 \ \forall j \in \mathbf{Z} \text{ or } \eta(2j + 1) = 0 \ \forall j \in \mathbf{Z}\}$. These are the configurations without a kink. It is customary to call configurations in this set *ordered phases*. Let $\mathbf{N}$ be the set of natural numbers and $\mathbf{N}_0$ the set of non-negative integers. We further define $E_1 = \{\eta | \ \eta[a, b] = (1, 0^{2k}, 1)$ for some $k \in \mathbf{N}_0$ and unique $a, b \in \mathbf{Z}$, $a < b\}$. Elements of $E_1$ contain exactly one kink (surrounded by two tails of ordered phases). The sets $E_i$, $i = 0, 1$, are invariant under the Rule 18.

Let $\alpha$ be the Bernoulli$(1/2, 1/2)$ measure on $E_1$ (i.e. on the unspecified bits of their ordered phase tails). In **[EN]** it was proved that given an $\alpha$-distributed initial configuration the kink in it performs a random walk with i.i.d. delay times. The walk has zero mean and its asymptotic squared variation equals to $n$. Hence a Brownian motion can be obtained from it via a standard scaling limit.

It is known that the pointmass on the all-zero configuration and the even Bernoulli distribution (together with their convex combinations) are the only invariant measures for the Rule 90 (**[L]**). Furthermore they are the only invariant measures for Rule 18 on $E_0$ since these two rules agree on this set. Hence our choice of $\alpha$ is the only non-trivial stationary environment for the kink process. In the case of Bernoulli$(p, 1 - p)$, $p \neq 0, 1/2, 1$, initial condition the resulting random walk is a non-stationary stochastic process (walk in a temporally inhomogeneous medium). It should converge to the stationary walk in the Cesaro-sense (since the environment does by [L]).

In most of the following analysis we concentrate on configurations with exactly two kinks in it. Call the set of all such configurations $E_2$. It can be formally defined as above or with a language-oriented notation by requiring that the elements of this set are of the form

$$((\cdot,0)^*,1,0^{2l},1,(0,\cdot)^c,0^{2r},1,(0,\cdot)^*). \tag{1}$$

Here $c,l,r \geq 0$. As ususal $*$ denotes an arbitrary repetition of the expression.

A pair of kinks *annihilates* exactly when $\tau$ is applied to $\eta[n,n+2k+2] = (1,1,\{0,1\}^k,1), k \geq 0$. The center of the resulting configuration $\tau\eta[n-i,n+2k+2+j] = (1,\{0\}^{2k+1+i+j},1),\ i,j$ positive and odd, is the center of the now vanished kinks. This definition is useful in the context of the jump correlations. We also observe that although the kinks in the prescribed configurations annihilate only those with $k$ odd can be observed after the first iteration. This is simply because configurations with $k$ even do not have any preimages. Figure 1. illustrates an evolution from an initial condition of the form (1).

Analogously one could define sets $E_i,\ i > 2$, of configurations with exactly $i$ kinks. The sets $E_i,\ i \geq 2$, are not invariant due to the annihilations. It looks highly likely that $E_0$ and $E_1$ are the only nontrivial (in the sense of measure) invariant sets that have a fixed finite number of kinks in them.

We denote by $X^l(n)$ and $X^r(n)$ the locations of the left and right kink in the $n^{\text{th}}$ iterate. The order of the kinks can't change during the evolution. In the simulations we considered a finite toral lattice and there the order for all positive times is defined from the order in the initial condition in the obvious way. The notation $d(n) = X^r(n) - X^l(n)$ is used for the *difference process.* Clearly $d(n) \in \mathbf{N}_0$ for all $n \geq 0$ for which it is defined. Annihilation takes place at first $n > 0$ for which $d(n) = 0$. It is also useful to think $d \geq 0$ as a parameter that divides the set $E_2$ into *shells* of constant $d$.

## 2. Methods

To uncover the details of the joint motion of two kinks a number of computer simulations were performed. The lattice was chosen with the periodic boundary condition. After the number of iterations was decided (usually in the 50-200 range) the size of the lattice was determined from the condition that the backward space-time cones of the kinks could have only one-sided intersection (with a very high probability). This was done to simulate the case of two kinks colliding in the infinite lattice where the dependency between the kinks enters from between only.

The intial condition was of the form (1) with ordered phases of equal sizes at the ends. The unspecified bits in the ordered phase were chosen independently pseudo-random with even probability.

The programs were written in Mathematica and most of them were run on a IBM RS/6000 workstation.

## 3. Spatial aspects of the joint motion

### 3.1. The difference process

We first present the statistics of the difference process $d(n) = X^r(n) - X^l(n)$. This is an integer-valued process that dies at the first hit at zero i.e. annihilation.

Let us consider the results of a typical simulation of this process. It consisted of 300 histories of 100 iterates each of a kink pair that was originally at distance 15 from each other. The lattice was 417 cells wide. The run resulted in 10775 moves and 94 of the histories were terminated in an annihilation.

Figure 2a. illustrates the jumps of $d(n)$. When at least one of the kinks expanded at time $n$ the entry $(d(n+1) - d(n), d(n))$ was incremented. The resulting cumulative distribution is plotted. $d(n)$ peaks at 15 as expected and the distribution is almost symmetric with respect to the vertical axis past this $d$-value. For smaller values the attractive motion is restricted and the asymmetry in the distribution is an indication of this. We note that the empty wedge between 10 and 11 o'clock is indeed always there and in the form shown. In Figure 2b. we have a magnification of the central area of Figure 2a. The grayscale has been altered to enhance the hem of the distribution.

The annihilations are prominent in both pictures since they constitute the entries on the diagonal line $d(n) = -d(n+1) + d(n),\ n \geq 3$. They indeed occur only at $d$ values $3 + 4m,\ m \geq 0$ as claimed. The distribution varies with $d$ and doesn't seem monotone or unimodal. This was confirmed in a number of runs with varying $d(0)$. To investigate this phenomenon the quotient of annihilations to the total number of moves from shell at $d$ was computed. These findings are plotted in Figure 3. All samples during the first five iterates were ignored to supress the influence of initial condition. Here $d(0)$ was uniformly distributed on odd numbers between 5 and 53. Note that annihilations at distance 15 seem more frequent than at 11. This was a consistently observed and reveals an interesting non-uniformity in the structure of the underlying state transition graph. The backward trees rooted at these two sinks have different growth rates. Hence their basins of attraction are of different size as well.

An attempt was made to analyse the state transition graphs of the action of the automata on $\tilde{E}_2$. Here $\tilde{E}_2$ is a reduction of $E_2$ where only equivalence classes of configurations are identified. These classes are defined by requiring the same two kinks with identical configurations between them. The action of $\tau$ naturally defines a directed graph on this node set. Again $d$ is a natural parameter to partition the set $\tilde{E}_2$. $\tilde{E}_2$ consists of 4 states on the shell 3, 10 on shell 5, 22 on 7 etc. The structure of these was determined exactly but due to intershell transitions this isn't enough to explain the indicated non-uniformity. Transitions to shell 3 only come from outer shells and this transition probability can be estimated using frequency of annihilations from figure 3. Shell 5 turns out to be repelling; on it transitions to outer shells are clearly favored. Consequently most $d = 3$ annihilations were not reached via a path visiting shell 5 immediately before the annihilation.

Finally we note that the subgraphs of nodes with large $d$ have a high degree of uniformity. Since the transition probabilities of the (independent) random walk on the entire graph are exactly known we expect that it can be further analysed.

### 3.2. Correlations

In order to measure the degree of loss of independence the correlation coefficient for the jumps of the two kinks, $r(X^l(\cdot+1)-X^l(\cdot), X^r(\cdot+1), X^r(\cdot))$, was calculated cumulatively as a function of the shell parameter $d$.

Figure 4a. illustrates the results. 250 runs of length 100 each were performed at each odd distance from 5 to 53. The sample size was not sufficient outside the interval from 4 to 60 for the correlation coefficient estimate to be statistically reliable.

The contribution from annihilations is again clear at distances 7, 11 and 15. The repulsion of shell 5 amounts to a small negative correlation. One also discovers that correlations at even distances are almost zero. This is a consequence of the intrinsic difference between even and odd shells. Kinks can expand simultaneously only if they are at an odd distance. Of course one expects this variability in statistics in the moves to be supressed with larger sample sizes for all but the smallest values of $d$.

Apart from annihilations the moves themselves are remarkably uncorrelated at all distances as shown in Figure 4b. (which is compiled from Figure 4a. run).

### 4. Temporal aspects

To further analyse the cumulative effects of the dependency on the motion of the kinks three different exit times were recorded.

We call $T$ an ***annihilation time*** if it is the smallest integer $n > 0$ such that $d(n) = 0$. By $T_{top}$ we denote an ***top exit time*** i.e. the smallest integer $n > 0$ such that $d(n) \geq 2d(0)$. Finally a ***pure annihilation time***, $T_p$, is an annihilation time which is not preceeded by a top exit.

Let $Y_t = B_t^{(1)} - B_t^{(2)}$ where $B^{(i)}$ are independent Brownian motions starting at $y_0$ and 0 respectively. Then $Y$ is a Brownian motion and $Y_t$ has a normal distribution with mean zero and variance $2t$. The density of the first hitting time, $T_B$, of this process to zero is given by ([**KT**])

$$f(t|Y_0 = y_0) = \frac{y_0}{\sqrt{4\pi t^3}} e^{-\frac{y_0^2}{4t}}. \tag{2}$$

If the kinks were independent a distribution for $T$ close to $f$ with $y_0 = d(0)$ would be observed.

Simulations consisting of 40000 runs of length 50 (or less if an annihilation occurred) were performed. Let us consider the one where the initial distance was 7. Some 24336 annihilations were observed. The mass annihilations (due to the form of the initial condition) observed at $n = 1, 3$ and 4 accurately correspond to the theoretical values obtained from analysing the shell 7 state transition graph. The specific structure of the graph in shell 7 rules the annihilations upto eight iterates. However after this a picture remarkably coherent with the independent case emerges. The Figure 5. shows $T$- and $T_B$-distributions superimposed after a least squares fitting of the vertical scales has been done (this just amounts to equaling the volumes of the annihilations). The data has been smoothened by averaging each time instant over its two immediate neighbors and both the data and the graph have been shifted seven steps to the left. Consequently the time range from 8 to 49 is shown. From the data one is tempted to guess that $d(n)$ has its variance slightly below $2n$. However more extensive simulations are needed to establish this empirically.

We note that this fit points towards a very strong loss of memory property of the process $d(\cdot)$. The data seems to be indicating almost identical statistical properties in these two systems even at late annihilation times. Yet we know that almost all annihilations take place at distances 3 or 7. If an annihilation occurs at time 40 at distance 3 the backward cones of the kinks have at least a 95% overlap and the initial conditions agree at least to 97.5%. The corresponding numbers in the case $d = 7$ are 86% and 92.5%. So even when the kinks have almost identical pasts an independent-like behavior prevails.

By definition the variables $T_{top}$ and $T_p$ have identical distribution if the underlying process has independent and symmetric transition probabilities (like $Y$ has). So it is of interest to know how these distributions look for our system. They were computed in the same runs as $T$ and they are superimposed (without scaling) in Figure 6. We again note a similar type of decay ($O(n^{-3/2})$). But due to mass annihilations in the small range ($n \leq 4$) and absence of top exits there we find a slight domination of $T_{top}$ later. Once annihilations at the first step (our initial conditions contained two (1,1)-kinks) are removed the remaining totals of annihilations and top exits are within 6% of each other. When the annihilations at steps 3 and 4 are taken into account the apparent imbalance in Figure 6 is explained. Hence the difference between the distributions of pure annihilations and top exits is significiant only up to seventh iterate. Overall this simulation confirms the earlier results that the combinatorial restrictions to moves in the small shells perturbs the left-right-symmetry of $d$. This phenomenon explains the early dominance of pure annihilations.

## 5. Discussion

In this study we find clear evidence that the randomness dominating the motion of a single kink breaks down in the interaction of the kinks. Hence the diffusion approximation that can be made exact in the case of a single kink ([**EN**]) needs to be treated with care in the case of a general initial configuration i.e. in the presence of many kinks. However the break down doesn't seem to alter the statistics in an essential way.

It seems natural to characterize the interaction between two kinks within two realms. At close range, $d \leq 15$, the combinatorial structure of the problem imposes restrictions to the moves with $d(n+1) - d(n) < 0$ but not to moves with $d(n+1) - d(n) \geq 0$. This leads to asymmetry in the jump distribution. In the $d > 15$ range the kinks seem to be moving as if almost independent. In both ranges various statistics of the joint motion agree with those of the independent random walks with unit variances. An exception is the $d \leq 7$ range. But even there the motion resembles the independent case in such a way that the total volumes of annihilations and top exits over all shells agree with a good degree of accuracy. This is crucial since that guarantees identical annihilation rates with the independent random walk (Brownian motion) case. It remains to be seen if the indicated division can be used to characterize the recurrence properties of the kinks.

We note that Lind's conjectures follow immediately if the kinks are independent annihilating random walks. In the case of nearby kinks the independence is clearly violated but the joint motion of two kinks closely approximates the independent case in the various statistical senses that we tested. And since multiple annihilations are extremely rare the case of two kinks considered here is indeed the key to the asymptotic dynamics. Our findings indicate nothing to refute the conjectures. Moreover any non-local property of the system i.e. any property that doesn't solely depend on the very close range interaction seems well approximable by the independent annihilating random walk model.

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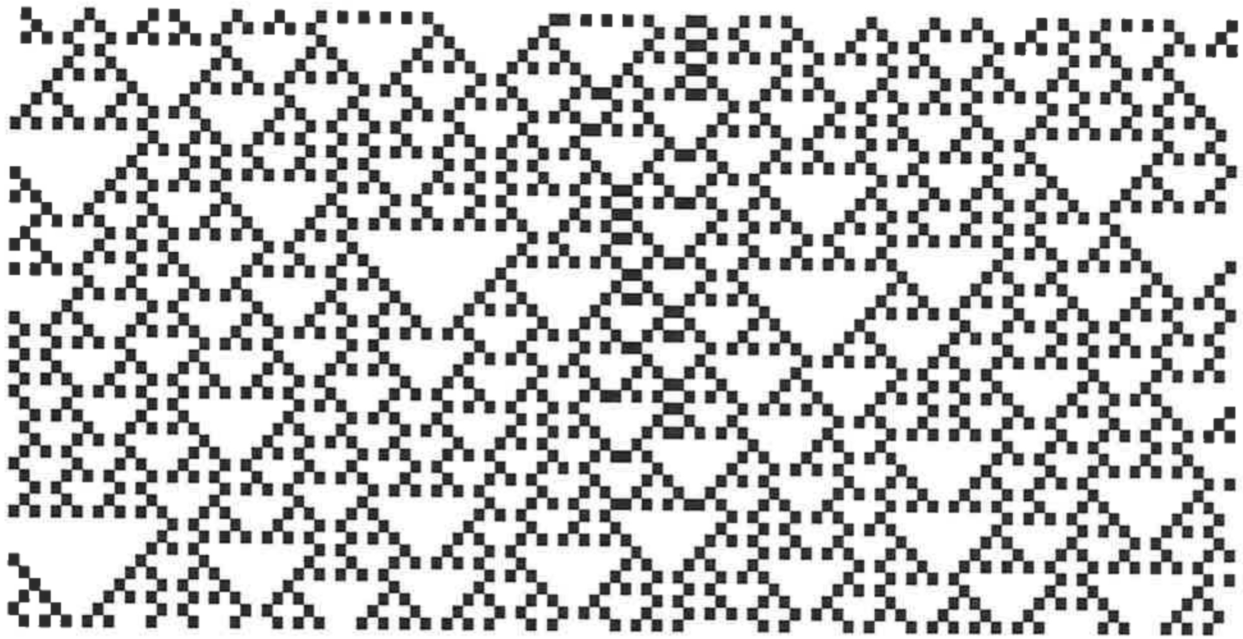

Figure 1. Annihilation of two kinks (at step 41, time runs downwards).

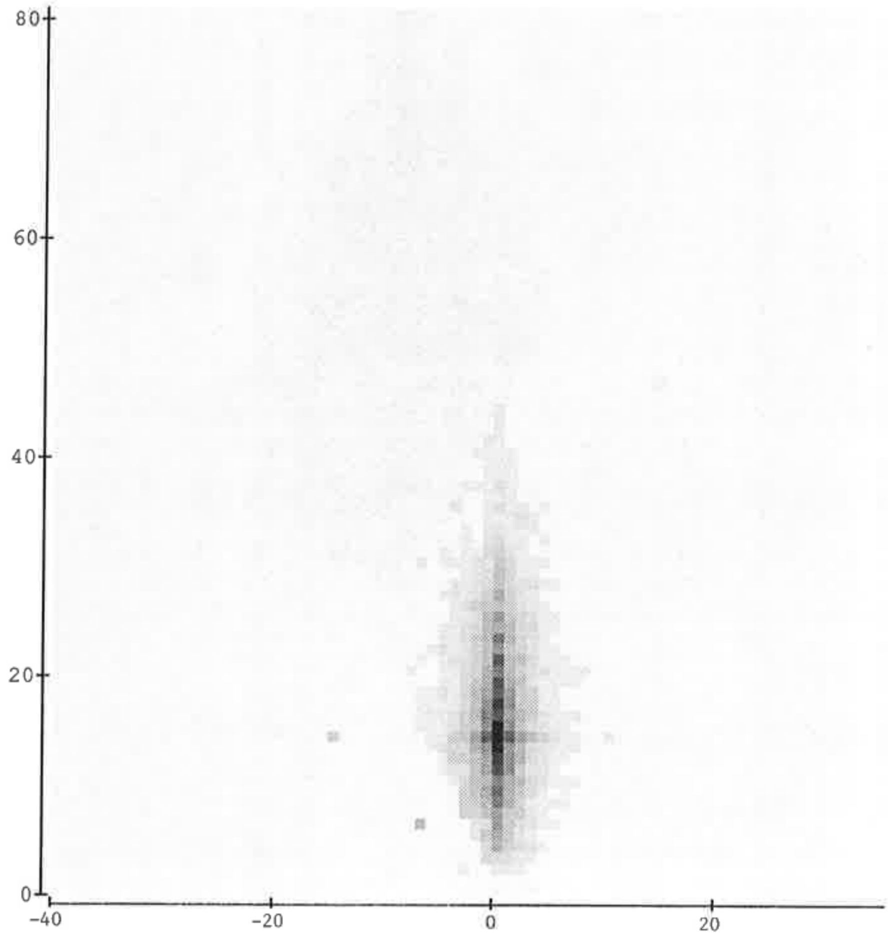


Figure 2a. Distance versus jump size.

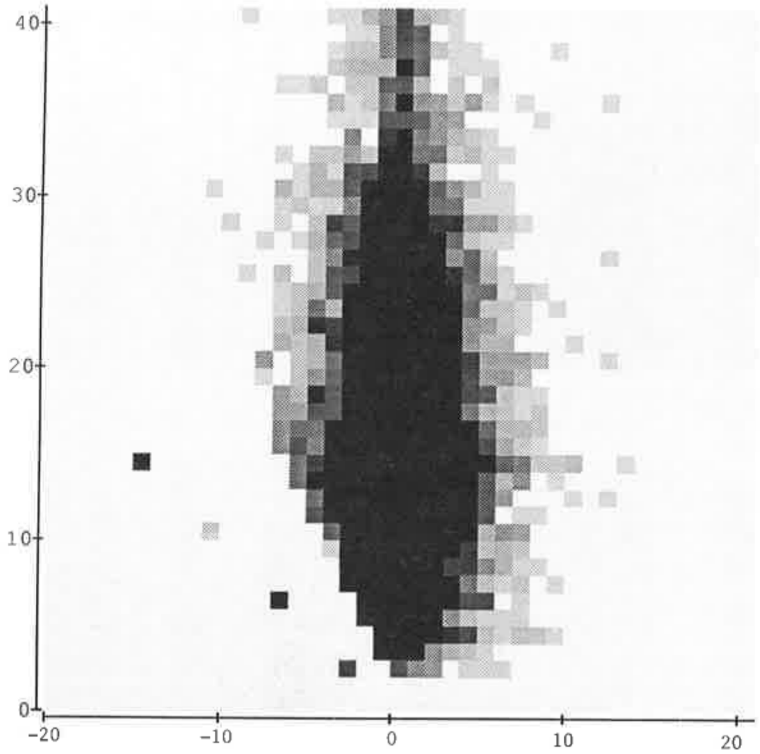


Figure 2b. Distance versus jump size.

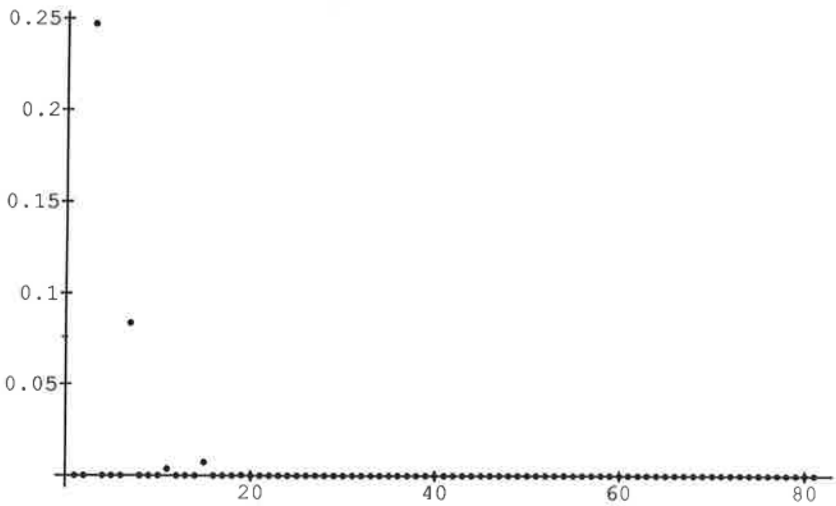


Figure 3. Annihilations/moves.

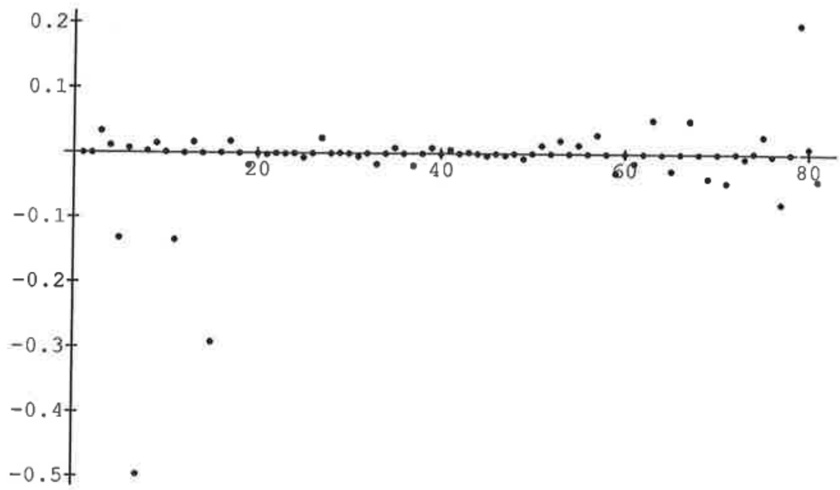


Figure 4a. Correlations.

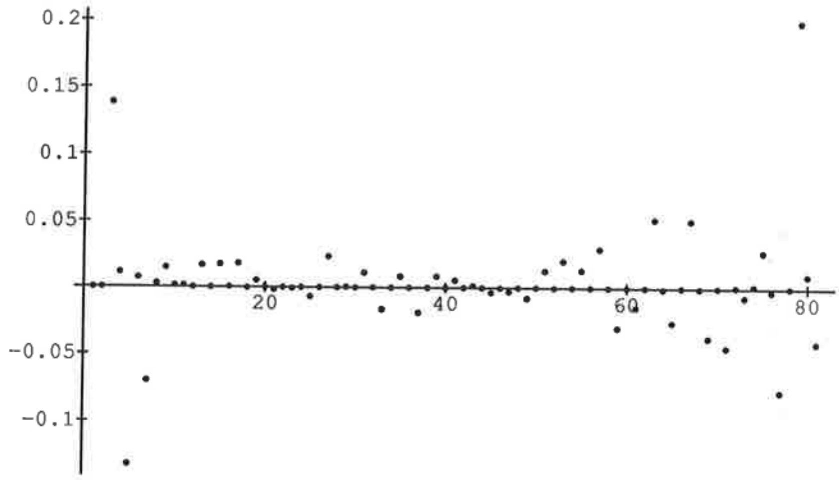


Figure 4b. Correlations, annihilations excluded.

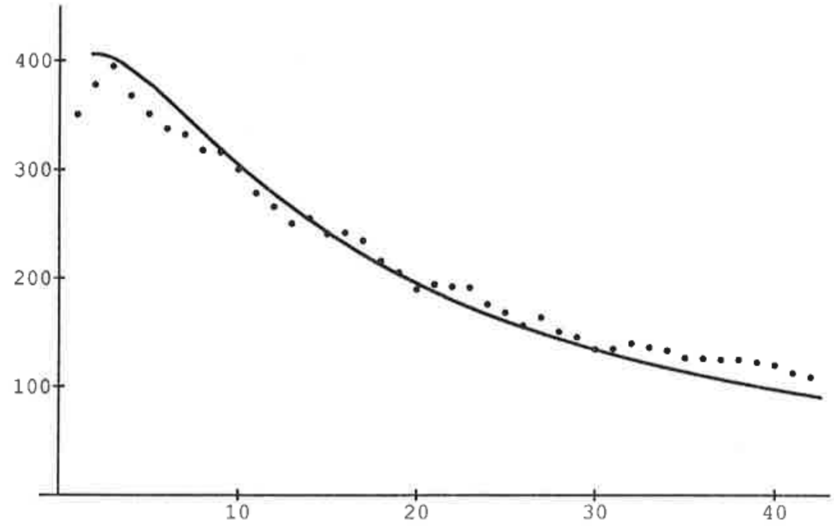


Figure 5. T- and $T_B$-distributions.

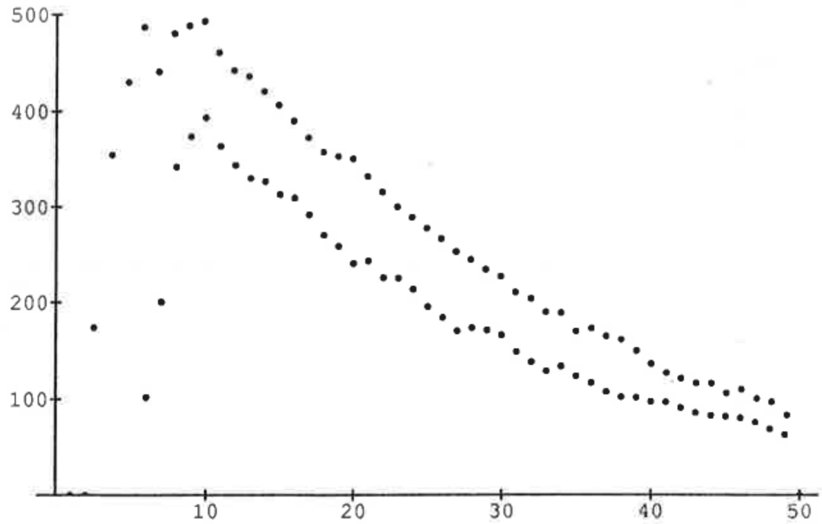


Figure 6. $T_p$- and $T_{top}$-distributions.