

Partially permutive cellular automata

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Abstract. One-dimensional cellular automata are analysed via their generalized permutivity. Invariant subalphabets provide a systematic way of identifying periodic and aperiodic tilings as well as stationary distributions invariant under the cellular automaton iteration. In the case of several invariant subalphabets a hierarchy of interaction phenomena arise. In particular the interaction of subalphabets can generate random walks as well as their degenerate forms. A comprehensive scheme emerges that unifies the analysis of topological defects in cellular automata. The probabilistic details of the random walks involved are treated in the companion paper in this issue.

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Introduction

Cellular automata are spatially and temporally discrete models of interacting dynamics. The interactions are local, yet can be made to mimic a wide variety of different physical phenomena. The theoretical motivation for the study of these systems is based on their close relation to lattice spin models in classical statistical mechanics as well as to the modern study of symbolic dynamics and coding theory.

In spin systems like the Ising model one can discern distinct phases separated by topological defects. They are also called Bloch walls, contours, dislocations or kinks. Under rather general conditions they perform random motions which in turn determine the macroscopic properties of the medium. This connection is established by applying to an ensemble of kinks a Peierls-type argument yielding estimates for quantities like entropy. Since a number of cellular automata (c.a.) also exhibit kink-like structures one hopes that an analogous analysis could be carried through in this context.

In this and the companion paper [4] we concentrate on the first part of this program. We characterize the motion of a single kink and the interaction spectrum of an ensemble of kinks in a large class of c.a. Our hope of course is that as in the classical case here too the ensemble of kinks will ultimately be easier to analyse than the plain model. Only one-dimensional c.a. are considered here but not because the underlying ideas are restricted to just them. The class of c.a. that we study are those with partial permutivity properties. Since we have encountered only a few c.a. which support kinks yet do not belong to this class we have reason to believe that partial permutivity is the principal mechanism behind diffusive kink motion and explains some of the soliton behaviour as well. In view of our results the random walk models for the kinks in c.a. are justified but their interaction phenomena require further investigation.

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Of course this work is not without predecessors. For a comprehensive and up to date treatment of random walks in modern statistical physics the reader is referred to [6]. Some of the closely related crystalline order formation and symbolic dynamics ideas are highlighted in [16]. The seminal empirical studies on kinks in c.a. by Grassberger [7] have been later extended and elaborated e.g. in [18] and [1]. A recent contribution to the subject relating it to general automata theory is [9]. Although c.a. were invented by mathematicians von Neuman and Ulam their mathematical analysis was initiated by Hedlund and his co-workers in [8]. Rigorous studies on substitution dynamics and blocking transformations related to our study are e.g. [2, 10].

The study of c.a. has no doubt been critically influenced by computer simulation of various interesting rules. We do not make any effort to cover this up by trying to 'axiomatize' the subject—indeed, we believe that it is more stimulating to reveal the nature of the subject through important case studies. For this reason we have presented only a few results formulated to theorems and the rest to rigorously studied examples only sacrificing some generality.

The paper is organized as follows. In section 1 we present the basic definitions and immediate consequences of partial permutivity in terms of tilings. This in turn enables us to recast certain old results on stationary distributions into a form useful to us. In the third section the consequences of subalphabet/tiling identification are investigated in the order of increasing complexity of the set-up starting from a pair of inert subalphabets and concluding with the general case of where annihilating, coalescing and branching random walks can occur. We finish in section 4 by considering some further extensions.

The division of labour between this and the companion paper [4] is simple. There we concentrate on the probabilistic aspects which can be completely worked out in the case of a single inert boundary between permutive phases. The general architecture of the class under consideration and the rest of the results are here.

1. Basic definitions and properties

We first present some definitions. In doing this we try to conform with established concepts and notation when possible. The basic reference here is Hedlund's paper [8].

Let $S = \{0, 1, \dots, |S| - 1\}$ be a finite *alphabet* i.e. the set of *symbols* and $X = S^{\mathbb{Z}}$ be the set of *configurations*. Let $\sigma : X \rightarrow X$ be the *left shift*: $\sigma(x)_j = x_{j+1} \forall j$. A $(m - n + 1)$ -*block map* is a map $f : (x_{j+n}, \dots, x_{j+m}) \rightarrow S$, $n < 0 < m$ redefining the value of the coordinate x_j .

Definition 1.1. One-dimensional cellular automaton (c.a.) is a dynamical system on X defined by a blockmap which commutes with the shift.

The map from X to itself defined by requiring $F(x)_j = f(x_{j+n}, \dots, x_{j+m})$ for all j is continuous (in the usual product topology) and conversely any continuous map $F : X \rightarrow X$ that commutes with the shift is induced by a block map [8]. If the block map satisfies $-n = m = r$ we call r its radius (r is integer or half-integer). From this on we restrict to this case although most results can be easily generalized. The case $r = 1, |S| = 2$ corresponds to the elementary cellular automata (e.c.a.). The block map f is also called the *rule* of the automaton. For elementary c.a. we use a Greek letter symbol for the rule. When the argument list $\{x_1, \dots, x_{n+m+1}\}$ can be left-right reversed without affecting the value of the block map the rule is symmetric.

Among the cellular automata particularly attractive are the permutive ones.

Definition 1.2. The n -block map is right permutive (in the sense of Hedlund) if $f(x_1, \dots, x_{n-1}, x_n)$ is a permutation for every fixed $(x_1, \dots, x_{n-1})$ -vector. A left permutive map is defined in a symmetric fashion. The map is permutive if it is both left and right permutive.

Example 1.1. There are 16 different maps on binary doubles. Two them are permutive, 6/16: 01 and 10 $\mapsto$ 1 and 9/16: 00 and 11 $\mapsto$ 1. Here as well as later in the paper we use the natural numbering of rules so for example $6(/16) = 2^{01_2} + 2^{10_2}$ (this is analogous to the numbering of e.c.a. e.g. in [18]).

The above definition of permutivity cannot be effectively extended to other entries than the extreme left and right ones of the block map. By this we mean that the key surjectivity result (proposition 1.2.) is obtained only when a distal entry permutes. This motivates us to redefine the c.a. in a way that makes permutivity a more natural concept and moreover immediately suggests ways of generalizing it.

Definition 1.3. Given the set of $4r$ -tuples of symbols in S a block map f of radius r defines a mapping from it to $2r$ -tuples. On the new alphabet $T = \{0, 1, \dots, |S|^{2r} - 1\}$ we thus obtain an induced cellular automaton map $\tilde{f}$ on two-blocks: $\tilde{f} : T \times T \rightarrow T$.

Note that the blocking introduced here does not need to preserve the symmetry of the rule unless a permutation in the $2r$ -blocks is also accounted. Two two-block maps induced by rules of radii 1 and $3/2$ are shown in figure 1. i is the iteration index (time, runs downwards from this on). In the subsequent analysis we primarily deal with the two-block representation and it secondary whether it was obtained via a tiling or not. Therefore the radius of the induced two-block rule is usually taken to be $1/2$ and it is then understood that the global c.a. map alternates between configurations on integers and half-integers.

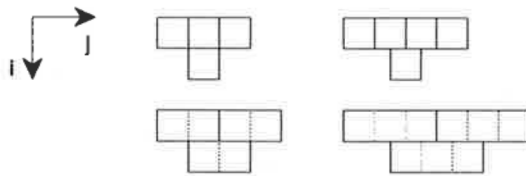


Figure 1.

A graphic consequence of this definition is that the spacetime evolution of the rule is converted into a tiling. The tiles are the $(2r \times 1)$ -rectangles that are piled in the fashion indicated in the figure.

Proposition 1.1. If the original block map was (right/left-) permutive (in the sense of Hedlund) so is the induced two-block map and conversely.

Proof. Pick $t^{(i)} = (t_1^{(i)}, \dots, t_{2r}^{(i)}) \in T$, $i = 1, 2, 3$, let $t^{(2)} \neq t^{(3)}$ and suppose that they agree upto the k th coordinate, some $k < 2r$. If the $(2r + 1)$ -block map is right-permutive then f applied to $t^{(1)}t^{(2)}$ and $t^{(1)}t^{(3)}$ cannot match beyond the k th entry. So $\tilde{f}(t^{(1)}, \cdot)$ is injective (and surjective).

Conversely if f is not right-permutive there is a $2r$ -vector and distinct $s_1, s_2 \in S$ such that $f(t_1, \dots, t_{2r}, s_i)$ agree. But then augment from the left the vectors $(t_1, \dots, t_{2r}, s_i)$ to two $4r$ -vectors. $\tilde{f}$ on these agree so its permutivity fails. $\square$

The two-block map allows a simple algebraic formulation of c.a. since we are given a finite set of symbols T closed under a binary operation $\tilde{f}$ between any two of them. If in the equation $\tilde{f}(t_1, t_2) = t_3$ between symbols from T any two determine uniquely the third one the system $(T, \tilde{f})$ is called a *quasigroup*. If it moreover has a identity element it is called a *loop*. Hence an alphabet with a permutive two-block c.a. map is a quasigroup or a loop. Note that these are not necessarily groups since associativity is not assumed and this would indeed be unusual for a c.a. action. They are nevertheless rare since they are the latin squares on the given alphabet. For a strictly subpermutive c.a. the entire alphabet together with the two-block rule is not a quasigroup but it has subsets which are (or are even loops). Note also that the identity, if it exists, is unique and forms a subalphabet by itself but it does not need to be contained in every permutive subalphabet. If a rule has a *quiescent state*, i.e. a symbol $s \in S$ such that $f(s, s, \dots, s) = s$ then of course the two-block rule fixes the appropriate symbol $t = (s, \dots, s)$ and $\tilde{f}(t, t) = t$ holds. The symbol t is a natural candidate for the identity.

The Cayley table will provide a compact representation of a rule. As an example we have included in figure 2 the table of the Klein four-group and a sample of the c.a. evolution corresponding to it. On the right we have the multiplication table of the elementary cellular automaton rule 18.

	0	1	2	3
0	0	1	2	3
1	1	0	3	2
2	2	3	0	1
3	3	2	1	0

0	1	1	2	2	0	3	3	3	1	0	2	3	2
1	0	3	0	2	3	0	0	2	1	2	1	1	
2	3	3	2	1	3	0	2	3	3	3	0		
3	2	0	1	3	2	3	2	1	0	0	3		

	0	1	2	3
0	0	1	2	2
1	1	0	0	0
2	2	3	0	0
3	1	0	0	0

Figure 2.

The Klein four-group is actually the Cayley table of one of the four permutive e.c.a., the rule 90. By inspecting the Cayley tables of the others one immediately notices that the rules 90 and 165 are groups and 105 and 150 are asymmetric quasigroups. Moreover the first two are a conjugate pair i.e. $(p \circ f_{90})(s, s') \equiv f_{165}(p(s), p(s'))$ for some permutation p on the four symbols and so are the last two. But by the stated asymmetry e.g. 105 cannot be conjugate to either 90 or 165.

Permutive maps have a number of nice properties. We briefly list here two essential for us. The following simple but important result is proved in [8].

Proposition 1.2. *A (left/right-) permutive cellular automaton is surjective.*

Since in a (two-sided) permutive rule the extreme entries of the $2r + 1$ -block permute the image the forward cone of influence is maximally wide i.e. a perturbation propagates at a maximal rate (r sites at one iterate). When appropriately formulated this can be viewed as permutive c.a. having maximal Lyapunov exponents among all rules of a given radius.

Permutive cellular automata even in their one sided form are quite rare. For example out of the 16 rules on binary doubles two are permutive and of the 256 elementary cellular automata there are 4 permutive and 24 left- or right-permutive ones. However permutivity prevails in a partial form in many rules and can still dominate the behaviour of the rule as we will see. We now proceed to formulate partial permutivity and its measures for two-block rules.

Let f be a two-block map on an alphabet T . Suppose that T_l and T_r are subsets of T such that $f(l, \cdot)$ and $f(\cdot, r)$ are permutations for each $l \in T_l$ and $r \in T_r$ respectively. If $p_l = |T_l|/|T|$ and $p_r = |T_r|/|T|$ then the rule f is called (p_l, p_r) -permutive. If $p_l = p_r = p$ it is simply called p -permutive. This definition however does not have useful closure properties and we therefore proceed to refine. By a maximal subalphabet with respect to P we mean a subset of the full alphabet with a property P such that if this subalphabet is augmented with any element from its complement it loses the property.

Definition 1.4. A subset T_r is a right-invariant subalphabet of T if $f(r, T_r) = T_r$, $\forall r \in T_r$ i.e. $f(r, \cdot)$ is right permuting on T_r for each $r \in T_r$. Left-invariant subalphabets are defined in a symmetric way. If T_l and T_r are maximal such subalphabets and $p_l = |T_l|/|T|$ and $p_r = |T_r|/|T|$ the rule is called (p_l, p_r) -subpermutive. If these maximal subalphabets coincide then $p_l = p_r = p$ and the rule is called p -subpermutive.

Remark 1. A left-invariant subalphabet does not need to be right-invariant and even if T_l and T_r are unique they are in general different. For a symmetric rule they of course agree.

Remark 2. Left-permutive cellular automata are clearly $(1, \cdot)$ -permutive as well as $(1, \cdot)$ -subpermutive and permutive ones are 1-permutive and 1-subpermutive. p_l and p_r parametrize the departure from surjectivity. Intuitively their non-trivial values should indicate subtle mixing/information transmission properties. Non-trivial values of p seem to be intimately related to high regular language complexity of the configurations generated by the automaton. This empirical observation was in fact a major impetus for this work. Another attempt to parametrize the space of cellular automata, Langton's λ -parameter [12], is related to our scheme. In the context of two-block rules λ is the fraction of two-blocks being mapped to a non-quiescent state (note that λ may be different for the original and the induced rule). With a little algebra one finds that if the rule is p -permutive then

$$p \left(1 - \frac{1}{|T|} \right) \leq \lambda \leq 1 - \frac{p}{|T|}$$

and it is p -subpermutive then

$$\max \left\{ 0, p \left(p - \frac{1}{|T|} \right) \right\} \leq \lambda \leq 1 - \frac{p}{|T|} \quad \text{or} \quad p^2 \leq \lambda \leq 1$$

depending on whether the quiescent symbol is in the maximal invariant subalphabet or not. For large alphabets and near permutivity the inequalities get tighter.

Given a subalphabet $T' \subset T$ the subset $X' = (T')^{\mathbb{Z}} \subset X = T^{\mathbb{Z}}$ consisting of sequences from this subalphabet is said to be *generated* by T' or *tiled* by T' -blocks if these are induced from a wider block map. By the surjectivity if the subalphabet T' is either left or right invariant then the image of X' under the c.a. is also generated by T' .

Definition 1.5. Let the collection of the left-invariant subalphabets of T be $\{L_i\}$. Define $q_l = \sqrt{|\bigcup \{(l_1, l_2) | (l_1, l_2) \in L_i \times L_i \text{ for some } i\}|} / |T|$ and q_r analogously for the right-invariant subalphabets. Then the rule is totally (q_l, q_r) -subpermutive.

The definition accounts for multiple permutive subalphabets. A symmetric rule which is p -subpermutive is totally q -subpermutive for some $q \in [p, 1]$. Note that since the subalphabets can overlap $q_l \leq \min\{1, \sqrt{\sum |L_i|^2 / |S|}\}$ holds. Definition 1.5 just amounts to obtaining the root of the fraction of the entries in the Cayley table belonging to some (left/right-) permutive subalphabet.

Before studying some more general rules we illustrate the use of these concepts in the context of perhaps the most investigated class of c.a.

Example 1.2. The elementary cellular automaton rule 18 is defined by requiring that 001 and 100 $\mapsto$ 1 while the other binary triples map to zero. It induces a two-block map on symbols $\{0, 1, 2, 3\}$ which has the permuting subalphabets are $T_1 = \{0, 1\}$ and $T_2 = \{0, 2\}$. This can be readily seen from the Cayley table in figure 2(b). The action of the induced c.a. map on the subalphabets is that of 6/16 on binary doubles (see example 1.1).

Example 1.3. The e.c.a. rule 22 is defined by requiring that 001, 010 and 100 $\mapsto$ 1 (call the block map τ_{22}). The induced two-block map only has an invariant symbol (00) which generates the quiescent state. The square of the rule, τ_{22}^2 , is of radius two and the two-block map it induces on the alphabet $\{0, 1, \dots, 15\}$ is 1/8-permuting. The six maximal (left- and right-) invariant subalphabets are $\{0, 1\}$ and $\{0, 7\}$ i.e. tile sets $\{0000, 0001\}$ and $\{0000, 0111\}$ together with their translates (as originally observed in [7]). The action on the subalphabets is again 6/16. Higher powers of τ_{22} yield families of larger tiles which seem however to be just piles of these.

Table 1. Table of partial permutivities of some elementary cellular automata.

Rule	Permutivity	Subpermutivity	Total subpermutivity
18	0	1/2	≈ 0.66
22 (1)	(1/2, 1/2)	1/4	1/4
22 (2)	0	1/8	≈ 0.29
30	(1, 0)	(1, 1/4)	(1, 1/4)
54 (1)	1/4	1/4	1/4
54 (2)	0	(1/8, 1/8)	$\approx (0.24, 0.24)$
90	1	1	1
122	1/4	1/2	0.61
126	0	1/2	1/2
146	1/4	1/2	≈ 0.71
170	(0, 1)	(1/4, 1)	(1/2, 1)
184	0	(1/2, 1/2)	$\approx (0.71, 0.71)$
204	0	1/4	≈ 0.35

In table 1 the list for each rule is up to the power (in the parenthesis) where the first non-trivial invariant subalphabet shows up if such exists. The rule 170 is the left shift and 204 is identity, others are non-trivial rules. The table is by no means exhausting. For example the rule 110 has partial permutivity properties but they are non-trivial only at such high powers of the rule that we could not find an exact characterization. The dominant period-block, a 14-tuple, is found as an invariant symbol for the two-block map corresponding to τ_{110}^{14} . In general there seems to be no bound to the power which needs to be analysed to find all interesting invariant subalphabets.

2. Invariant measures and tilings

The existence of an invariant subalphabet provides us with an invariant measure i.e. a probability distribution on the configuration space that remains invariant under the c.a. iteration. We now formulate this and examine the invariant configurations.

Recall that by proposition 1.2. the (left/right) permuting c.a. are surjective. These automata preserve the uniform product measure, i.e. Bernoulli measure with even probabilities independently on each lattice site. Suppose that f is a two-block map and F the global c.a. map and that the superindex 1/2 refers to the lattice of half-integers.

Proposition 2.1. *A cellular automaton F with a left/right invariant subalphabet $T' \subset T$ preserves the uniform Bernoulli measure $\mu_{T'}$ i.e. $\mu_{T'}^{(1/2)} = \mu_{T'} \circ F^{-1}$.*

Since $\mu_{T'}$ is a product measure it is of course preserved by the shift σ . Hence if one views the c.a. evolution (space-time history) as a $\mathbb{Z}^2$ -action the shift to any direction is measure preserving on configurations generated from T' .

When $T' = \{t\}$ the invariant measure $\mu_{T'}$ is just a point mass concentrated on the homogeneous state. If f was induced from a rule of radius larger than $1/2$ then t corresponds to a tile and the c.a. preserves a periodic tiling of the plane.

In case T' has multiple elements the invariant configurations generated from it are almost surely aperiodic in the sense of the measure $\mu_{T'}$ (this is just because almost every sequence in the support of $\mu_{T'}$ is aperiodic).

Since both aperiodic and periodic tilings invariant under a c.a. iteration are of interest by themselves we also want to point out their construction. A given two-block rule may not have a non-trivial invariant subalphabet yet some power of it may induce a two-block rule with this property hence implying the existence of an invariant periodic tiling. An illustration of this is the rule in example 1.1. which is permutive on $\{0, 1\}$ but only has a trivial proper invariant subalphabet $\{0\}$. However e.g. $T' = \{011011\}$ is invariant under the two-block map induced by the fourth power hence a periodic tiling exists. Indeed it is fairly easy to show that an infinite number of distinct invariant periodic tilings exist for this rule.

3. Interaction of the subalphabets

The interaction of two subalphabets/tilings can lead to a variety of different phenomena. We first consider the characteristics of an individual interaction boundary and thereafter the interactions in the case of multiple boundaries. The aim is to present the spectrum of possible behaviour in partially permutive c.a. in a unified way.

Definition 3.1. *Given two subalphabets S_1 and S_2 let $A = S_1 \cap S_2$ be the set of ambiguous symbols. If it is non-empty it is by itself an invariant subalphabet. Ambiguous symbols are receding i.e. $f(s, a) \in S_i \setminus A$ for all $s \in S_i \setminus A$, $a \in A$ and $i = 1, 2$. The configurations in the set $\underline{S_1 S_2}(j) = \{s_k \mid s_k \in S_1 \forall k \leq j, s_k \in S_2 \forall k > j \text{ and } s_j, s_{j+1} \notin A\}$ are said to have a boundary point at $j + 1/2$.*

The set of ambiguous symbols is usually empty or consists of one element which is generated from the quiescent state. Note that if the subalphabets are disjoint ($A = \emptyset$) then every configuration in $\underline{S_1 S_2}$ (i.e. left half generated from S_1 and right from S_2) is in $\underline{S_1 S_2}(j)$ for a unique j . If $A \neq \emptyset$ then any configuration of the form $\underline{S_1 A S_2}$ where $\underline{A}$ is a finite block of symbols from A is eventually reduced to the form $\underline{S_1 S_2}$ (the $\underline{A}$ -block is shortened by one at each iterate of the automata hence the number of iterates needed to this reduction is $|\underline{A}|$). Therefore the definition above applies again and we define the location of the boundary point in between these instances by interpolating.

In physics terminology we just identified the phase boundary/topological defect/kink/dislocation between two ground states.

3.1. Two inert subalphabets

The basic case is the one where the collection of sets $S_1 S_2(j)$ is closed under a c.a. map. In this *inert* case the boundary will prevail at all times and no new ones are created.

Example 3.1.1. In example 1.2. we identified the subalphabets $T_1 = \{0, 1\}$ and $T_2 = \{0, 2\}$. Their boundary defines a kink and the action of the rule 6/16 (induced by the elementary rule 18) is inert on $T_1 T_2$. Ordered in the reverse direction the action is not inert anymore since symbol $3 = 11_2$ is produced but the multiplication table (figure 2, right) is of course closed. Here the ordering is not essential in characterizing the motion of the boundary point. This case was analysed in [5] where it was proved that the phase boundary performs an unbiased random walk.

If in the table the shaded element is changed to 0 we have a perfectly symmetric (and closed) c.a. action on a subalphabet and a unbiased random walk will prevail under iteration. However if instead we have 1 or 2 in this entry then the configurations $T_2 T_1$ show T_1 or T_2 respectively dominating i.e. winning all the interactions not involving the neutral element 0. As a result the phase boundary moves in a monotone fashion at maximal speed (1/2) either to the left or right.

The two distinctly different types of motion that a boundary point can have are those of a *signal* and a *random walk*. By a signal we mean rectilinear motion with maximal speed as in the example above. Upon creation the motion is completely deterministic. Hence it is only dependent on a finite part of the initial configuration. On the other hand random walks are motions which exhibit strong dependency to the initial configuration. If this is distributed according to the appropriate product measure the boundary motion will perform a non-deterministic, stationary motion and have positive variance etc. It can have independent increments and hence be Markovian but this is not the case in general. Signals can be viewed as degenerate forms of random walks (zero variance). For the exact definitions and the dichotomy between these types we refer to [4].

When the motion of the boundary point is considered for a c.a. with a random initial configuration random walks prevail under fairly general circumstances if neither of the subalphabets is dominating with respect to the other one. This and some other features of the interaction can be readily read from the Cayley table. We now proceed to define some concepts that are useful at characterizing the absolute and relative strength of an invariant subalphabet. For simplicity we restrict the definitions to the symmetric case and the straightforward but index rich extension to the general case is left to the reader.

Definition 3.1.1. Let $S_1 \subset S$ be a permutive subalphabet and $N = \{e\}$ or $\emptyset$ depending whether S has an identity or not. If f is a symmetric two-block map on $S \times S$ we define the first basin of attraction for the symbol $i \in S_1$ to be the set $B_i^{(1)} = \{s \in S \mid f(i, s) \in S_1 \setminus N\}$ and the first basin of attraction of S_1 to be $U^{(1)}(S_1) = \bigcap_{i \in S_1 \setminus N} B_i^{(1)}$. Similarly let the second basin of attraction for the symbol $i \in S_1$ be $B_i^{(2)} = \{s \in S \mid f(i, s) \in U^{(1)}(S_1)\}$ and the second basin of attraction of S_1 be $U^{(2)}(S_1) = \bigcap_{i \in S_1 \setminus N} B_i^{(2)}$. The symbol i is dominant over S if $B_i^{(1)} = S$. If all $i \in S_1 \setminus N$ are dominant then S_1 is. If $S_2 \subset S$ is another permutive subalphabet and $A = S_1 \cap S_2$ let $N = A \cup \{e\}$. Then the first/second basin of attraction for the symbol i /subalphabet S_1 with respect to S_1 is defined as above with the new N and S replaced by S_2 .

The following useful result holds when the identity (if exists) is not in the subalphabet.

Proposition 3.1.1. If S_1 and N be as above and $S_1 \cap N = \emptyset$ hold. Then $S_1 \subset U^{(1)}(S_1) \subset B_i^{(1)} \subset B_i^{(2)} \forall i \in S_1$ and $U^{(1)}(S_1) \subset U^{(2)}(S_1) (*)$.

Proof. By $S_1 \subset B_i^{(1)}$ and the definition of $U^{(1)}(S_1)$ the first two inclusions hold. And $S_1 \subset U^{(1)}(S_1)$ applied to the definitions of $B_i^{(1)}$ give the third one which implies $U^{(1)}(S_1) \subset U^{(2)}(S_1)$. $\square$

Theorem 3.1.2. If $S_1 \cap N = \emptyset$ and

- (i) $U^{(1)}(S_1) = S_1$ then $U^{(2)}(S_1) = S_1$,
- (ii) $U^{(1)}(S_1) = S$ then $U^{(2)}(S_1) = S$ (S_1 dominates over S),
- (iii) $U^{(2)}(S_1) = S$ then two adjacent symbols from S_1 generate a column of width at least one.
- (iv) If $e \in S_1$ and $U^{(1)}(S_1) \cap S_1^c = S_1^c$ then S_1 dominates ($S_1^c = S \setminus S_1$).

Proof. (i) and (ii) follow directly from the definitions. For (iii) we notice that two adjacent entries of S_1 guarantee an element in S_1 in the next iterate. By the assumption next to it we will have an element in $U^{(1)}(S_1)$. So there will be at least two adjacent S_1 symbols in the second iterate. If $s \in S_1 \setminus e$ and $U^{(1)}(S_1)$ is all of S_1^c then $f(s, \cdot) \in S_1 \setminus e$ and (iv) holds. $\square$

Remark 1. Part (iv) of course complements the dominance in the $S_1 \cap N = \emptyset$ case (ii). Note however that if $e \in S_1$ then $S_1 \not\subset U^{(1)}(S_1)$ i.e. we cannot have $U^{(1)}(S_1) = S$.

Remark 2. Part (iii) refers to the (dominationwise) intermediate case where a vertical barrier exists. It may not expand (a strict inclusion in $(*)$ in proposition 3.1.1) but it cannot vanish and will hence block the motion of other boundary points. The existence of a barrier limits the possibilities for the drift direction of a signal or a random walk.

These results state of course just sufficient conditions. Even the existence of just one dominant symbol may result in maximal expansion as seen in example 3.1.1. Clearly at most one dominant subalphabet exists and if it is found in the initial configuration it guarantees that at any given site the action at sufficiently large iterates is permutive.

Example 3.1.2. In figure 3(a) we have a Cayley table for a c.a. with two permutive subalphabets $S_1 = \{1, 2\}$ and $S_2 = \{3, 4\}$. Now $N = \emptyset$ and $U^{(1)}(S_1) = \{1, 2, 3\}$, $U^{(2)}(S_1) = S$ and $U^{(1)}(S_2) = U^{(2)}(S_2) = S_2$. Hence a barrier exists. In the variant in 3(b) (we substitute for the shaded block as well as its symmetric counterpart) $U^{(1)}(S_1) = S_1$ and $U^{(1)}(S_2) = \{2, 3, 4\}$. But the symbol 1 is dominating i.e. wins over all of S_2 and the same phenomenon prevails as in the case of an expanding subalphabet. In 3c $U^{(1)}(S_1) = S_1$ and $U^{(1)}(S_2) = S_2$. This rule supports a symmetric random walk on the boundary between S_1 and S_2 which is analysed in [4]. The three rules are 0-permutive, 1/2-subpermutive and totally 1/2-subpermutive.

		1	2	3	4			0	1	2	3
(a)	S_1	1	1	2	1	3	(b)	0	0	0	0
	2	2	1	1	1	3		1	1	1	2
	3	1	1	3	4			2	2	2	2
	4	3	3	4	3			3	1	1	2
							(c)				

Figure 3.

When the subalphabets are small $U^{(1)}$ cannot be complicated. But in the asymmetric case things can still be rather interesting.

Example 3.1.3. A particularly intriguing case of signals dominating the evolution of a c.a. is seen in the context of the elementary c.a. rule 184 (empirically motivated e.g. in [11] and studied in [17]). See table 1 for its permutivity characteristics and figure 3(d) for the Cayley table. All the definitions and results above can here be applied in their one-sided form.

The two-block map of τ_{184} is identity on each of the symbols $\{0, 1, 2, 3\}$ i.e. each of these generates an invariant configuration. Moreover it has two right- and two left-invariant subalphabets, $\{0, 1\}$, $\{0, 2\}$ and $\{1, 3\}$, $\{2, 3\}$ respectively. The actions of $f(\cdot, 0)$ and $f(\cdot, 1)$ restricted to the first set are identities. So both 0 and 1 dominate over each other from the left and a right propagating signal results. The second set generates the same signal and the last two similar left moving ones. Due to this structure the signals appear in pairs bounding a few 11_2 's between checkerboard patterns generated by the tilings made of $1 = 01_2$ and $2 = 10_2$ (in the case of the last two sets).

Example 3.1.4. In the context of the elementary c.a. rule 54 we have another example of a signal. In this case the only permutive subalphabets found consist of a single element and they are for τ_{54}^2 (higher powers seem to generate just multiples of these). Two of them, 4 and 7 in their natural numbering, are illustrated in figure 4. (the others are shifts of these). Each of them generates the same invariant configuration under τ_{54}^4 . In fact it is a attracting tiling in the sense that asymptotically it seems to prevail and form the background for all the action. The Cayley tables of 4 and 7 as well as their mirror images 2 and 14 are shown. These show dominant symbols and reveal the left and right propagating signal. The evolution on the right illustrates the signal i.e. the crack between two perfect tilings generated by 4's and 7's.

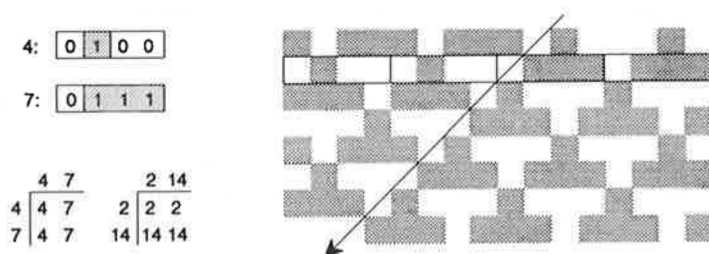


Figure 4.

By a simple parity check one sees that upon collision two signals merge to form a stationary symmetric structure. This in turn vanishes when reflecting off a third signal. Generating the left half of an initial configuration from 2 and 14 independently with identical probabilities and the right half similarly from 4 and 7 this structure recurs with a positive frequency and performs an unbiased random walk. This random walk is however not generic. But the rule supports at least two generic kink-like structures which interact with the signals, can create them and which can pairwise annihilate. The characterization of the motion of even the simpler one of these structures, although random like, seems complicated. The joint motion of the kinks is highly dependent due to the exchange of signals (see [1]).

As we can see in these examples the size of the subalphabets dictates whether features like barriers can exist. Also for large alphabets one intuitively expects a subalphabet which has an overrepresentation in the crossterms of the Cayley table and hence large $U^{(i)}$ to dominate the behaviour of the boundary in some fashion. However this may be transient behaviour even for high density—if the other subalphabet has a single dominant symbol it may eventually prevail. Some subtle questions remain which are formulated and answered exactly in [4] where the equilibrium behaviour is characterized.

3.2. Several inert subalphabets

When the initial configuration is not required to consist of just two distinct semi-infinite phases several boundary points can be found. If the boundaries are sufficiently far apart (so that the backward cones of the boundary points do not intersect) the motion of an individual point under the c.a. iteration is independent of the others and is as described in the previous section. But their (close range) interaction brings new elements into the set-up. We proceed by presenting a simple example which already has the germs of all inert interactions in it.

Example 3.2.1. On the top left of figure 5 we have a Cayley table for a c.a. with three invariant subalphabets $S_i = \{0, i\}$, $i = 1, 2, 3$. Next to it we have an example of a nine generation evolution. Suppose that in the figure $x = 1$, i.e. that the initial configuration was generated by just S_1 and S_2 . The shaded lines indicate the boundary lines between the tilings. We observe that at the seventh iterate the subalphabet S_2 becomes extinct i.e. only S_1 is observed from that on. Equivalently the boundaries *annihilate* and a single phase S_1 emerges. However if $x = 3$ then the boundaries *merge* at the critical iterate and a single boundary remains (hence the shaded boundary line). Note that since our subalphabets are permutive and in fact identical to the one's considered in example 3.1.1 we know that individually the boundary lines should be performing random walks if each S_i -block was generated from Bernoulli measure. So apart from a possible dependency of the motions our example seems to indicate the presence of annihilating and coalescing random walks respectively.

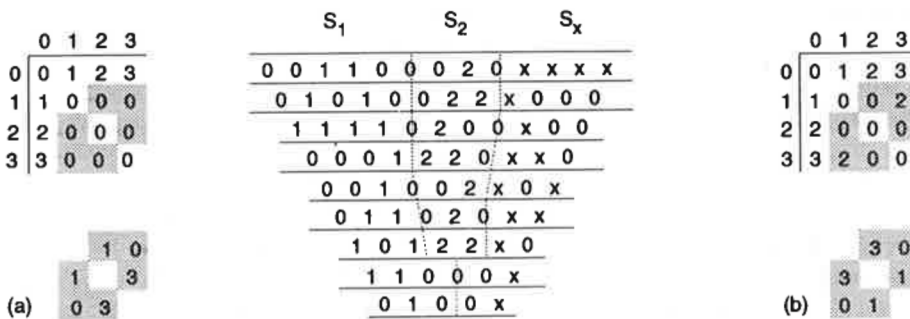


Figure 5.

Note that in these cases the crossterms in the Cayley table all equal to the identity element 0. Any other (symmetric) choice would lead to either a dominating subalphabet or creation of new subalphabets from interaction. As an example of the former we have indicated an other rule obtained by substituting to the Cayley table by the shaded elements below it. In this rule both S_1 and S_3 dominate over S_2 . If the phases are ordered as before

for the initial configuration we again obtain two boundary points but this time moving to the right and left by $1/2$ step each iterate i.e. signals. Upon collision of the boundary points S_2 becomes extinct. However the remaining tilings have identical attraction properties and the newly formed boundary thereby performs an unbiased random walk (characterized in [4]).

We also point out that these annihilation and coalescence phenomena can take place at arbitrary distance but only between neighbouring boundary points. In the case of two subalphabets the former is possible only when $A = S_1 \cap S_2 \neq \emptyset$ (as seen in our example with $x = 1$). Analogously if neighbouring phases are generated by intersecting alphabets coalescing at distance may occur (our $x = 3$ -case).

In the same vein one can present a simple c.a. with three subalphabets exhibiting the coalescence of random walks into a signal. By considering larger alphabets biased random walks together with degenerate variants can be introduced as explained in the previous section. These can coexist and in fact in particular examples their interaction algebra i.e. rules how the boundaries recover upon collision can be worked out much the same way as the tiling algebras represented by the Cayley tables (see the last section).

In the absence of births the density (frequency) of kinks is non-increasing in time due to annihilations/coalescings. In fact it seems that generically arbitrarily long blocks of pure phase are eventually formed. This leads to the questions of the existence of an attractor and the type of convergence towards it. In [13] Lind formulates conjectures on the attraction in the context of aforementioned elementary c.a. rule 18 on the basis of observed asymptotics of the evolution. Computer simulations seem to indicate remarkable closeness to independence even at close range [3]. The conjectures would follow from the independence of the boundary motions immediately. The source and degree of the independence in a large class of c.a. subsuming this special case is characterized in our upcoming work.

3.3. Potent subalphabets: branching random walks

In the interactions considered so far the number of boundary points has been non-increasing since no creation of new boundaries have been allowed. However if the subalphabets are not inert symbols of new subalphabets may be born out of interaction and we can reach the full generality. Again we present the main cases in simple examples.

Example 3.3.1. If the top Cayley table in figure 5(a) is modified by introducing a non-zero element in the shaded block we can arrive to the top table in figure 5(b). In this rule subalphabet S_2 interacts inertly with the others whereas S_1 and S_3 can parent S_2 . When this occurs the boundary branches into two. So in addition to the annihilation and coalescing behaviour we also see *branching*. The rule obtained by replacing the shaded elements by the ones below is twice as potent—only S_2 is never born out of interaction. This rule supports the full spectrum of interaction types as well.

Example 3.3.2. Suppose we have a c.a. with the Cayley table as in figure 6. x is any of the four symbols. Clearly phases S_1 and S_2 , S_1 and S_4 , S_2 and S_3 as well as S_3 and S_4 exhibit signals at their mutual boundaries. Upon collision *soliton-like* transmission of signals prevails as indicated in the middle illustration. In order to make this interpretation we regard the distinguishing feature of a signal to be its direction, not the types of the phases bordering to it.

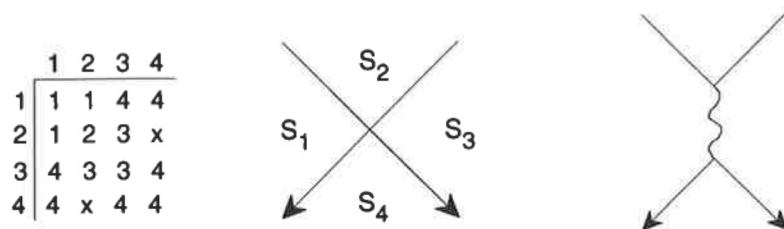


Figure 6.

If the subalphabets S_1 and S_3 are not singletons then of course the interaction between them can result in a random walk. This still resembles the soliton behaviour if the lifetime of the $S_1 S_3$ -boundary is short as is typically the case when S_4 is predominant among the $S_1 S_3$ -cross-terms (figure on the right). If the expected lifetimes of both diffusive- and soliton-interactions are long we may encounter metastability-type phenomenon. This arises naturally in the context of large alphabets which partition into nearly invariant (rarely interacting) subsets. For elaboration see the end of [4].

Soliton-like behaviour has been found in more complicated c.a. earlier and it seems to be a dominant feature in the evolution of some related automata models as shown in [15]. Apart from the physical motivation coming from modelling solitary waves this behaviour may be important from the computational point of view. Solitons enable information transmission in one-dimensional c.a. critical in the construction of computational primitives like logic gates. By incorporating subalphabets with soliton characteristics one can build in this property in a subpermutive c.a.

The global mixing properties of some of the maximally branching cases are understood. Suppose that all the symbols belong to some permutive subalphabet. Furthermore define the cross terms in such a way that the full alphabet becomes a quasigroup. This cannot of course always be done. But when it is possible all the cross interactions result in a symbol in a third subalphabet, i.e. we are in the maximally branching case. Moreover now all results on permutive c.a. apply. This extension is illustrated in our last example if the two zeros in the shaded area are switched to twos. We then obtain the Cayley table of the Klein four-group (on the left in figure 2).

4. Extensions

In definition 3.1 we required that the permutive subalphabets border to each other. Sometimes we do have just one permutive subalphabet or two different ones which however are not immediate neighbours yet permutivity of the phases can still be utilized and similar boundary motions identified.

Example 4.1. A totalistic rule of radius r on binary $(2r+1)$ -tuples is defined by requiring that the update is 1 iff $0 < \sum_{i=1}^{2r+1} x_i < 2r+1$. For any $r \geq 1$ the rule has an invariant permutive subalphabet $T^{(r)}$ consisting of two symbols, all 0's and all 1's blocks of length $2r$. Consider for simplicity the case $r = 2$. Let $\underline{B}$ be a block of all zeros or all ones of length 1, 2 or 3. Then each of $\underline{T}^{(2)} \underline{B} \underline{T}^{(2)}$ with the usual uniform measure generates a distinct symmetric random walk. A boundary point can be identified as the centre of the non-tiling block and at any time this has a maximum length 5, 6 and 7 for the respective

kinks. Note that any configuration of 0's and 1's can be decomposed into a sequence of tiles and kink-blocks. In the presence of multiple kinks annihilation and coalescence occur due to change of parity in the non-tiling blocks and simple additive collision rules for the three kinks can be worked out (see [1]). Same analysis applies to all rules with $r \geq 1$. The attractor seems always to be $\underline{T}^{(r)}$.

The key here is that the length of the non-tiling block is short and remains so under the evolution. The permutive phases confine a thin boundary layer and its right and left jumps are generated as before.

Example 4.2. As we have seen in example 1.3 and in table 1 the notorious elementary cellular automaton rule 22 has partial permutivity properties. Let $T_2 = \{0000, 0010\}$ and $T_4 = \{0000, 0100\}$ be two of the invariant subalphabets (for τ_{22}^2). Then $\underline{T_2 T_4}$ (each part again Bernoulli distributed) exhibits a boundary point performing an unbiased random walk. This is analysed as rule 18 in [5]. The other subalphabets can be combined with similar result. Moreover we can produce statistically identical random walks in this rule from configurations of the form $\underline{T_2 B T_4}$ where B is a suitable finite block of 0's and 1's.

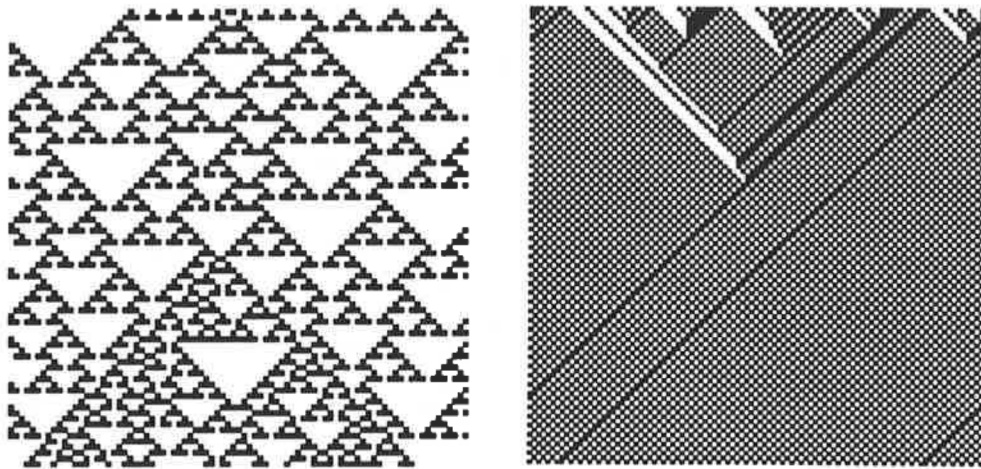


Figure 7.

On the left of figure 7 we have an 80-step evolution from $\underline{T_2 I I T_4}$ on a periodic lattice of width 82. Note that there are initially three kinks two of which merge at iterate 27 into a kink of another type. Although the kinks observed here are not of the type described in definition 3.1 (B is non-trivial) their motion can be rigorously analysed because they are confined between permutive phases and thereby the statistical properties of the left and right jumps are known (the walks have zero drift and unit variance four, see [4]). And in particular we note that unlike indicated in [7] these kinks when confined are stable. This particular B can be replaced by other blocks with similar results.

Unfortunately the permutive subalphabets will not dominate the behaviour of this rule. This is quite clear from the bottom half of the figure. The remaining two kinks give birth at iterate 43 to the rather unwieldy 'complex' phase. The same kinks can be found there, too but intertwined in a complicated way. The collision rules between different types of kinks can be worked out but obviously their ruletable is not closed.

Confining a block between two permutive tilings works for a number of other rules as well. A non-diffusive case arises in the context the rule 184 of example 3.1.3. An 80-step evolution from the uniform Bernoulli distribution is shown on the right in figure 7. We note that two neighbouring parallel signals interact with a signal pair travelling to the opposite direction in a simple way—the pair with larger gap prevails.

A particularly rich collection of rectilinear boundaries is known for the e.c.a. rule 110 (see e.g. [22, p 549]). Interestingly even some of those not moving at the maximal velocity ± 1 can be explained using the dominant invariant tiling as in example 3.1.4 (now a 14-block) together with the block confinement argument above. In particular the existence of 'signals' with velocities $-2/4$, $0/7$ and $2/3$ follows.

References

- [1] Boccara N, Nasser J and Roger M 1991 *Phys. Rev. A* **44** 866
- [2] Dekking F M and Keane M 1978 Mixing properties of substitutions *Z. Wahr.* **42** 23–33
- [3] Eloranta K 1991 The interaction dynamics of the kinks in the cellular automaton Rule 18 *Research Report A306* Helsinki University of Technology, Institute of Mathematics
- [4] Eloranta K 1993 Random walks in cellular automata *Nonlinearity* **6** 1025–36
- [5] Eloranta K and Nummelin E 1992 The kink of the elementary cellular automaton Rule 18 performs a random walk *J. Stat. Phys.* **69** No 5/6
- [6] Fernández R, Fröhlich J and Sokal A 1992 *Random Walks, Critical Phenomena, and Triviality in Quantum Field Theory* (Berlin: Springer)
- [7] Grassberger P 1983 New mechanism for deterministic diffusion *Phys. Rev. A* **28** 3666–7; 1984 Chaos and diffusion in deterministic cellular automata *Physica* **10D** 52–8
- [8] Hedlund G A 1969 Endomorphisms and automorphisms of the shift dynamical system *Math. Sys. Theor.* **3** 320–75
- [9] Hanson J E and Crutchfield J P 1992 The attractor-basin portrait of a cellular automaton *J. Stat. Phys.* **66** No 5/6
- [10] Jen E 1990 Aperiodicity in one-dimensional cellular automata *Physica* **45D** 3–18
- [11] Krug J and Spohn H 1988 *Phys. Rev. A* **38** 4271
- [12] Langton C 1986 Studying artificial life with cellular automata *Physica* **22D** 120–49
- [13] Lind D A 1984 Applications of ergodic theory and sofic systems to cellular automata *Physica* **10D** 36–44
- [14] Miyamoto M 1979 An equilibrium state for a 1-dimensional life game *J. Math. Kyoto* **19** No 3
- [15] Park J K, Steiglitz K and Thurston W P 1986 Soliton-like behaviour in automata *Physica* **19D** 423–32
- [16] Radin C 1991 Global order from local sources *Bull. Am. Math. Soc.* **25** No 2, 335–64
- [17] Smillie J 1987 Properties of the directed entropy function for cellular automata *Dynamical Systems, Proc. University of Maryland* ed J C Alexander pp 689–705
- [18] Wolfram, S. 1986 *Theory and Applications of Cellular Automata* (Singapore: World Scientific)

