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VOTER DYNAMICS IN DETERMINISTIC CELLULAR AUTOMATA

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ABSTRACT

A deterministic counterpart is introduced to the voter model studied in probabilistic particle systems. Here we investigate the ingredients of the rule as well as its annealing domain dynamics. It is a representative from a larger class of cellular automata with the common property that they all show behavior previously associated to lattice models with strong independence assumptions. The results indicate that purely deterministic schemes are capable of producing behavior characteristic to disordered systems of statistical mechanics.

INTRODUCTION

We will introduce design principles by which one can build up cellular automata (c.a.) rules with dynamical properties remarkably close to the probabilistic voter models (see e.g. Durrett). The motivation for this comes from the currently unsatisfactory level of understanding of the dynamical capabilities of c.a. In particular their statistical mechanics has been described in a number of simulation studies but the fundamental theory is missing. Our design principles not only create ideal c.a. in the sense of statistical mechanics but also sheds some light into why some earlier rules behave as they do (see e.g. Toffoli and Margolus).

The one-dimensional theory has been established in earlier work (Eloranta 1993 a,b,c). Here we will study the two-dimensional case where qualitatively new phenomena appear as a consequence of the neighborhood topology. This paper is a introduction to this new behavior – the work on a larger class of rules with annealing as well as critical dynamics will appear later.

The presentation is structured as follows. We first introduce a splitting of a c.a. rule which is performed to see how a stationary source of randomness can prevail in a deterministic rule. This structure ensures a maximal degree of mixing and facilitates a pseudorandom contour motion. Thereafter we determine the appropriate ("voter") neighborhood action and show why it implies the annealing dynamics.

1. THE MIXING STRUCTURE

Let $S = \{0, 1, 2, 3\}$ be a set of *symbols*, the *alphabet*. Let L be the square lattice $\mathbf{Z}^2$ and $L^{(1/2)}$ its dual lattice $(\mathbf{Z} + 1/2)^2$. The sets $X = S^L$ and $X^{(1/2)} = S^{L^{(1/2)}}$ are the sets of *configurations*. On both of these we have the natural coordinate actions, the *horizontal* and *vertical shifts* defined by $(\sigma_h x)_{(j_h, j_v)} = x_{(j_h+1, j_v)}$ and $(\sigma_v x)_{(j_h, j_v)} = x_{(j_h, j_v+1)}$ for $x \in X$.

Definition 1.1. The map on four symbols in a 2×2 square neighborhood $f : S^4 \to S$ is a block map or a cellular automaton rule. A cellular automaton is the map $F : X \to X^{(1/2)}$ and $F : X^{(1/2)} \to X$ obtained by requiring that the cellular automaton rule commutes with the shifts σ_h and σ_v.

To index the space-time configurations we will subsequently use j as spatial indices and the superindex i as temporal index indicating the iteration of the c.a.

The representation given in the Definition 1.1. is a particularly useful because it enables one to distinguish invariant sets of configurations.

Definition 1.2. The c.a. map is partially permutive if it has an invariant subalphabet: there exists $S' \subset S$ such that $f(s_1, s_2, s_3, S') = S'$ for any $s_i \in S'$ and the same equation holds for any permutation of the arguments in f.

The global configurations $S'^{\mathcal{L}}$ and $S'^{L^{(1/2)}}$ generated from an invariant subalphabet S' are invariant under F. It seems appropriate to call them *pure phases* since they play that physical role in our model as will be seen.

The symbol set S contains four symbols just for the purpose of defining the basic set-up. Moreover we assume that it partitions into two invariant subalphabets of equal size. To define the rules we represent the symbols in two parts. For each symbol we write $s = (a, d)$ where $a \in A = \{0, 1\}$ and $d \in D = \{0, 1\}$. A is the set of subalphabets and D is the set of *digits*. In order to see all possible subalphabet interaction phenomena we would need three of them but two suffices for the basic phenomena to appear. The set D has to be nontrivial for the c.a. to have a non-singular probabilistic structure as we will see.

To streamline the formalism let $\mathbf{s} = (s_1, s_2, s_3, s_4)$, $\mathbf{a} = (a_1, a_2, a_3, a_4)$ and $\mathbf{d} = (d_1, d_2, d_3, d_4)$ be the symbol, subalphabet and digit vectors in a 2×2 neighborhood (oriented as (NW, NE, SE, SW)). Our rules are of then of the form

$$f(\mathbf{s}) = \left(A^{(\mathbf{a})}(\mathbf{d}), Q(\mathbf{d})\right).$$

The table $\underline{A} = \left\{A^{(\mathbf{a})}\right\}$ is called the *assignment array*. As functions the arrays $\underline{A}$ and Q must map as $\underline{A}: S^4 \to A$ and $Q: D^4 \to D$.

The stationarity of the c.a. is at the level of the digit evolution. Suppose that Q is the map

$$Q: \mathbf{d} \to \sum_{i=1}^{4} d_i \pmod 2. \tag{1.1}$$

This map has the property that the global c.a. map P induced by it (Q is a c.a. rule on digits) is permutive and hence preserves the uniform Bernoulli measure: $\mu_D = \mu_D^{(1/2)} \circ P^{-1}$. Here μ_D and $\mu_D^{(1/2)}$ are these measures supported on $D^{\mathcal{L}}$ and $D^{L^{(1/2)}}$ respectively. The preservation follows directly from the fact that the map P is an endomorphism of the full shift i.e. shift-action on $D^{\mathcal{L}}$ (argued as the one-dimensional case, see Coven and Paul). Since we want a non-singular measure to be preserved we must exclude the case where D is a singleton.

(1.1) is a generalization of the basic one-dimensional rule introduced by Ledrappier. It is one of several rules providing the maximal degree of mixing in the digit evolution. Briefly it is described as follows. A finite sequence of measures μ_D and $\mu_D^{(1/2)}$ can be coupled to get a measure $\nu([0, i-1])$ on the cylinder set of i-step evolutions of P from 0 to $i-1$ starting from the invariant distribution. Using a standard extension argument this yields a measure ν on the set of bi-infinite evolutions of the c.a. P. Alternatively one view this as the construction of the inverse limit (see Petersen).

Let σ_t be the time-like shift i.e. shift in the direction of the c.a. action: $(\sigma_t x)_{(j_h, j_v)} = (Px)_{(j_h, j_v)}$ for all $x \in D^{L^{(1/2)}}$. From the work of Kitchens and Schmidt it follows that the joint action of the three shifts is asymptotically independent i.e.

$$\nu\left(\sigma_h^{j_h} \sigma_v^{j_v} \sigma_t^{i}(A) \cap B\right) \longrightarrow \nu(A)\nu(B)$$

as $|j_h| + |j_v| + |i| \to \infty$ for any measurable A and B.

The stationarity i.e. the σ_t-invariance of ν is all that we need from the above formulation in this paper. We do not utilize the mixing property directly but it is stated since our c.a. should be compared to a probabilistic voter scheme. We also note this is the maximal degree of mixing in the context of deterministic c.a. The shift in the time direction for such c.a. cannot have positive entropy hence independence is bound to be broken in a serious way in this direction. It is of interest to see if the remaining degree of independence is sufficient to provide us with dynamics qualitatively similar to that observed in probabilistic models that use heavy independence assumptions.

The preservation of the uniform Bernoulli measure on digits implies that a good initial measure μ for the F iteration is such that its marginal on the digits is μ_D or $\mu_D^{(1/2)}$. We assume this for the rest of the paper.

2.1. THE VOTER RULE

Having defined the underlying digit dynamics we now complete the definition of the rule. First we record the principles for the assignment arrays which hopefully make it more palatable from physical point of view (for a physicists approach to this type of models see Baxter). In doing this the necessity of the assumptions on the digit evolution will become clear.

The subalphabets represent our two voter populations or the + and −-phases in a physical medium. A homogenous 2×2 neighborhood supports its kind – this is the subalphabet invariance as in Definition 1.2. Hence the rule should preserve both of the subalphabets. On the boundary between two domains the majority opinion/phase in a 2×2 neighborhood should dominate in the determination of the update in the dual lattice. In case of an even 2-2-distribution of the subalphabets the assignment should be uniformly Bernoulli distributed at all times. This is where the stationarity enters the picture.

Furthermore we require that there is no preference in favor of either class in the dynamics and that the rule is isotropic on the square lattice L. The former requirement amounts formally to invariance under flipping the subalphabet assignment: $f \circ \iota = \iota \circ f$ where $\iota(a) = 1 - a$ whereas the latter corresponds to the rule being invariant under cyclic permutations of the arguments in the neighborhood vector (which we have ordered cyclically). Unfortunately the last two requirements can be simultaneously satisfied only when there is an uneven representation of the two subalphabets in a neighborhood. To see the problem in a $\mathbf{Z}^2$-neighborhood consider two arrangements of subalphabets, $\mathbf{a} = (0,0,1,1)$ and $\mathbf{a} = (1,1,0,0)$. An isotropic block map would give the same assignments in both cases whereas conjugacy invariant map would force a flip in the assignment. In our rule we impose conjugacy invariance first. Moreover isotropy can be recaptured in an average form as seen below. Note also that the rule will automatically be isotropic on digits since the update depends only on the sum $\sum_1^4 d_i$.

The assignment array of the *majority voter rule* is defined as follows. By even representation we mean that the two subalphabets both have two symbols in the neighborhood.

$$A^{(\mathbf{a})}(\mathbf{d}) = \begin{cases} a, & \text{if } a_i = a \text{ for at least three } i\text{'s} \\ a_1 + \sum_{i=1}^4 d_i \pmod 2 & \text{at even times if even representation} \\ 1 - a_1 + \sum_{i=1}^4 d_i \pmod 2 & \text{at odd times if even representation} \end{cases}$$

By the first part of the definition the action on the subalphabets is permutive and the majority domination explains the name. In the case of even representation of subalphabets in the neighborhood the checkerboard labeling given is just one of the possibilities. The key property that it has is that flipping of any single digit flips the labeling as well. Note that by our assumption that the measure on digits is always uniform Bernoulli both of the subalphabets have probability $1/2$ of winning the update in the case of even representation.

If we would have simply $a_1 + \sum_{i=1}^4 d_i \pmod 2$ in the even update the rule would still produce assignments in the two subalphabets with equal probabilities. But only two different symbols would be possible for any even $\mathbf{a}$. The given rule produces all of the four symbols with equal probabilities and consequently will not have bias to any direction. It is isotropic in this weaker sense. Both the given and the simpler rule are conjugacy invariant.

2.2. BASIC PROPERTIES OF THE RULE

In the subsequent analysis we think of each lattice (or dual-lattice) site having a 1×1-cell centered at it. This cell belongs to one of the subalphabets and in particular the boundaries between domains of different subalphabets are then defined.

Call a closed rectangle with sides parallel to the diagonals of $\mathbf{Z}^2$ ($y = \pm x$) a diamond. The *diamond hull* of a bounded set in the plane is the smallest diamond containing the set. So for example the diamond hull of four cells in a 2×2 arrangement centered at the origin is the set $|y| + |x| \leq 2$. If the set is unbounded then its diamond hull is a half space or -wedge defined in the obvious way.

The appeal of the majority voter rule is based on the following unique property.

Theorem 2.1. Given a domain generated from an invariant subalphabet suppose its diamond hull does not intersect the diamond hull of any other such domain. Then the domain will remain inside its diamond hull under the iteration of the majority voter c.a.

Proof. Let B be a domain generated from the subalphabet $S^{(0)}$ surrounded by a sea of symbols from $S^{(1)}$. Let $H(B)$ the the diamond hull of B. Suppose that we augment the set B by one cell $c \in H(B)$ i.e. flip the assignment of one of the $S^{(1)}$-symbols making it into a $S^{(0)}$-symbol. The number of corners the new cell has in common with B is between zero and four. By considering the action of the rule in a neighborhood centered at each of these corners one can see that the update in the augmented case is in $S^{(0)}$ whenever it is so in the original case (but not necessarily the converse). Hence $F(B) \subset F(B \cup \{c\})$ where F is the global c.a. map. Extending the set B to $H(B)$ cell by cell we get by induction that $F(B) \subset F(H(B))$. But the diamond hull clearly has the property that $F(H(B)) \subset H(B)$ and the conclusion follows.

Remarks 1. Note that no reference is made to the digit distribution in the initial measure and in particular the Bernoulliness of the digits is not assumed here. Indeed the result only hinges on the structure of the assignment array.
2. If the diamond hulls of two domains intersect then it is possible that the domains could merge. But again if the diamond hull of the union of the domains is isolated it is a confining diamond hull.

The Theorem clearly hints towards annealing dynamics. The random motion of the boundary of a bounded domain should eventually allow an even smaller diamond hull to be fitted around the domain. Hence any finite island should either merge to another one of the same type or as an isolated island vanish in finite time. Asymptotically we should see arbitrarily large homogenous domains. We now proceed to analyze the mechanisms behind this.

Suppose first that we have the lower half of the lattice $L^{(1/2)}$ below the diagonal $y = x$ generated from the subalphabet $S^{(1)}$ and the rest of the lattice sites form $S^{(0)}$. Denote the $S^{(a)}$-domains by $B^{(a)}$. If we now flip the assignments of the cells under the diagonal in the first quadrant we introduce a *boundary defect* at origin. Call the lines $L_l : \ y = x - 1$ and $L_r : \ y = x + 1$ the *left and right hull-lines*. Let L be the line parallel to them and at equal distance from both. So now L is the diagonal $y = x$. The defect is at the unique crossing point of L and $\partial B^{(\cdot)}$. We say that this defect has the *off-set* $L_r(0) - L_l(0) - 1 = +1$.

If we would reflect the domains with respect to $y = x$ the lines L_r and L_l would be swapped. The defect would still be at the origin but in this arrangement we define its off-set as $L_r(0) - L_l(0) + 1 = -1$.

It is clear that under the iteration of the majority rule the left and right hull-lines remain the same and so does the off-set of the defect. Moreover we have

Theorem 2.2. Suppose that the initial state is such that the digits are μ_D-distributed and that the assignments are as above i.e. that we have a boundary defect on the line $y = x$. Then the defect will almost surely perform a nearest neighbor random walk on the diagonal. Its location at the i^{th} iterate is (X_i, X_i) where $X_i = \sum_{k=1}^{i} \Delta_k$ and $\Delta_k = \pm 1/2$ with equal probabilities. The increments Δ_k are independent.

Proof. Suppose that the boundary defect is originally at (x_0, x_0). Its movement is uniquely determined once we know the assignment of the cell in the dual-lattice at the location (x_0, x_0). But this is determined by the symbols in the four cells cornering to the defect. In this neighborhood the representation of the subalphabets is even. Hence the digit at each of the cells permutes the digit in the center cell and hence flips the assignment at (x_0, x_0). So the increment is $\pm 1/2$ with equal probability. In the following iterates the mechanism is similar: assignment is again flipped by at least two independent digits at the corners of the backward cone. This is a pyramid with top at the defect and bottom a square in the initial state with sides aligned with the axis and centered at the top. Call the square at the i^{th} iterate P_i. Its side increases by 2 at each iterate thereby being $2i + 2$ after i iterates. The increment at an arbitrary iterate is determined by the entries in the square. It contains all the earlier squares (the past of the walk) and a frame of width one of new independent entries in the initial state. The corner digits on the decreasing diagonal of the square d_{nw} and d_{se} are independent of the past i.e. all the smaller squares inside P_i and independent of each other. The top is permuted by these digits and since it is always in a 2-2-neighborhood the increment of the defect is $\pm(1/2, 1/2)$ with probability $1/2$ independently of the past.

Quick combinatorics will show that if we have a $+1$-defect and a -1-defect on the diagonal $y = x$ they *annihilate* upon collision i.e. the left and right hull lines will be of (horizontal) distance one after that. In case of multiple defects the recombinations are always binary.

Suppose that we have an *alternating chain* of defects on the diagonal. By this we mean a chain where neighboring defects have a common hull-line and every $+1$-defect is between two -1-defects and viceversa except at the ends (in case there is only a finite number of defects).

Corollary 2.3. Suppose that we have an initial distribution where the digits are uniformly Bernoulli distributed and the assignment is such that we have an alternating chain on the diagonal $y = x$. Under the iteration of the c.a. the defects perform individually symmetric nearest neighbor random walks as in the Theorem 2.2. Moreover they are independent upto the time of collision with a neighboring defect which results in an annihilation. The defects annihilate each other maximally i.e. at most one is eventually left. If there is a positive initial density of defects on the line the density decays proportional to $1/\sqrt{i}$ where i is the iteration index.

Proof. The motion of an individual defect is argued as in the proof of the preceding Theorem. The independence of the increments of two defects at (X_i, X_i) and (X_i', X_i') is an consequence of the corner digits $d_{\rm nw}, d_{\rm se}$ and $d'_{\rm nw}, d'_{\rm se}$ in their corresponding squares P_i and P_i' being distinct and hence independent. Note that these digits are also independent of the pasts of both of the walks since the latter are contained as subsquares inside P_i and P_i'. The argument extends immediately to an arbitrary subcollection of defects in the alternating arrangement. So we can utilize the results for the case of independent random walks recorded e.g. in Griffeath's notes. The decay rate follows from there.

One can immediately extend the given formulation for any nonzero integral off-set. The motion of these defects can be argued analogously to Theorem 2.2. but now new phenomena enter if the off-set is not equal to ± 1 or in the case of just ± 1 off-sets we do not have the alternating chain.

Suppose that we have a single defect with off-set o, say $o \geq 2$ on the diagonal. Then under the iteration of the majority voter c.a. the defect may *branch* into two defects with off-sets o_1 and o_2 such that

$$o_1 + o_2 = o \quad (*)\,, \qquad\qquad o_1 o_2 \geq 1 \tag{2.1}$$

The off-sets here are counted as before from left and right hull lines of the boundary segments immediately to the left and right of the defect. The branching is reversible and the conservation of the off-set (2.1) $(*)$ also holds for the mergings. It is a immediate property of the majority voter rule that the offspring of a branching defect always has off-set with the same sign as the parent.

If the defects merging have off-sets of opposite sign (2.1) $(*)$ still holds. But now $o_1 o_2 \leq -1$ so their merging is *irreversible* by (2.1). A particular case of this is of course the annihilation considered earlier $(o_1 = -o_2)$.

One can summarize the implications of these principles in another topological statement.

Proposition 2.4 Suppose that we have a finite chain of defects with finite off-sets $\{o_n\}_{n=1}^{N}$. Denote the off-set sequence at the $i^{\rm th}$ iterate by $\left\{o_n^{(i)}\right\}_{n=1}^{N(i)}$. Then $o_i^{(i)}$ and $N(i)$ are uniformly bounded and the total off-set $\sum_{n=1}^{N(i)} o_n^{(i)}$ is constant and the total variation of the off-sets $\sum_{n=1}^{N(i)} |o_n^{(i)}|$ is nonincreasing.

The irreversibility of the mergings of defects with opposite off-sets is the very reason for the annealing behavior of this c.a. A curved boundary of a domain can be decomposed into a chain of defects with appropriate off-sets. Their irreversible mergings result in a monotone decrease of the total variation i.e. straightening of the boundary. In case of an isolated island this together with the random motion of the contour implies the eventual shrinkage of the confining diamond hull.

The difficulties in proving a global annealing result seem mostly just combinatorial. They are due to the technicalities in handling defects with arbitrary off-sets and their initiation at corners where hull-lines intersect. Higher order defects split constantly and since defects with different off-sets move at different rates the treated off-set ± 1-case seems to be the only clean one. The launching of defect pairs at corners where increasing and decreasing hull-lines intersect is somewhat complicated. Note however that the Theorem 2.1. tells that an isolated domain never escapes from such corner.

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