

Dynamical Vertex Models

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Introduction

1 Introduction

- The Ice Model
- Lattices
- Bounded case

2 Dynamics

- Cycles, boundaries, connectivity
- PCA

3 Order and disorder

- Subdomains, entropy
- Diamond squared
- Sectional probabilities
- Arctic geometry
- Demarcation on T/K

4 3.4.6.4.

- Extremal tilt, entropy
- No-go

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The Ice model

Let $\mathbf{G}$ be a connected graph without self loops and with even vertex degree. Orient each edge of $\mathbf{G}$ in the following way.

Definition

A **vertex configuration** is the arrangement of arrows arriving to and departing from a vertex. It is **legal** for the **Ice rule** if there is the same number of incoming and outgoing arrows. If there is a legal vertex configuration at every vertex of $\mathbf{G}$ the arrow configuration is legal for the **Ice model**.

On $\mathbf{G} = \mathbf{Z}^2$ Ice is also known as the **Six vertex model**. It is a remarkably good model for the physical ice. It's was analyzed by E. Lieb in the 60's.

Until recent years little was known about this model in a **bounded domain**.

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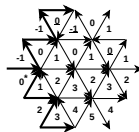
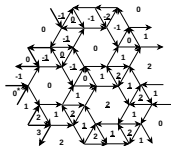
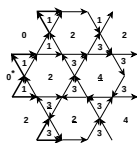
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Lattices

Archimedean/uniform lattices are the natural generalizations of the three regular lattices. They are formed as vertex and edge sets of the tilings of the plane by regular polygons. Of the 11 Archimedean lattices four support Ice: **square, triangular, Kagomé and 3.4.6.4. lattices** (6/20/18/36 vertex rules respectively).



Kagomé (**K**), 3.4.6.4. and triangular (**T**) lattices with height on the dual.

Bounded case: known unknowns

We will study these four vertex models in particular domains: the original square lattice Ice in a **diamond**, the other three models in a **hexagon** (with sides aligned with the underlying lattice).

Questions:

- When is a boundary arrangement of arrows legal i.e. allowing a fill-in? When does it allow **multiple fill-ins**? Exponential number of fill-ins?
- In case of multiple fill-ins, can the boundary influence the **deep interior**? If so, what is the structure there generically like? These are questions about the support of the measure of maximal entropy.
- Are the four vertex models behaving alike i.e. is there **universality**?

The investigation was initially motivated by the **Dimer/Domino model** studies by Propp et. al. related to crystal growth, faceting etc.

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Cycles

A basic observation: in an Ice configuration an unidirectional cycle can be reversed (infinite cycle, too, if existing).

Definition

*Depending on the lattice **1-triangle/1-square/1-lozenge/1-hexagon** denotes the smallest such polygon. If such **1-polygon** is unidirectional we call it a **1-cycle** and the reversal of this orientation a **local move/flip**.*

Later we will also need to keep track of the geometric orientation of the triangles and lozenges, but let's keep it simple now.

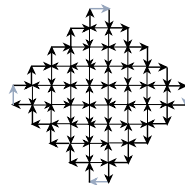
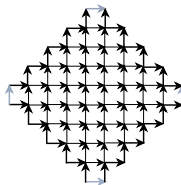
Proposition

An unidirectional cycle always encloses a 1-cycle.

Proof is a simple argument utilizing the flux across the cycle.

Boundary cosets

A cycle is **off-boundary** if it doesn't contain any of the boundary arrows. A configuration is **frozen** if it is without off-boundary 1-cycles. A **temperate** configuration has a directed cycle boundary.



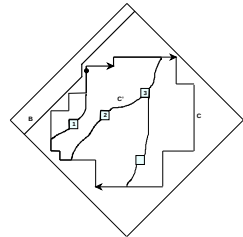
Basic question: Given a fixed boundary, a frozen configuration cannot be deformed but what about the others?

Connectivity

Theorem

The set of Ice configurations with common boundary arrows in a diamond (for square lattice) and in a hexagon (the other three Archimedean lattices) is connected under 1-cycle reversals i.e. two such configurations can be transformed to each other with a finite sequence of 1-cycle reversals.

Two lines of proof: 1. **lexicographic sweep** “fixes” the difference between two configurations with identical boundaries site by site from the boundary on.
2. **height argument** converts each local height minimum to a maximum eventually connecting the configurations via a maximal configuration, all with the given boundary configuration.

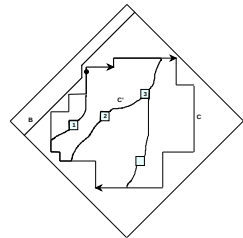


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Connectivity, computation

Through counterexamples one establishes

Refinement

If for triangular/Kagomé/3.4.6.4 lattice any one of the 2/3/6 types of local moves is disallowed the Theorem fails.

Computing Ice

1. Divide the arrow configurations into **arrays of oriented polygons**:

- $\mathbb{Z}^2$: checkerboard: black and white/even and odd arrays of 1-squares.
- **T**: two arrays of 1-triangles ($\triangle$ and ∇).
- **K**: two 1-triangle and one 1-hexagon arrays.
- **3.4.6.4**: two 1-triangle, one 1-hexagon and three 1-lozenge arrays (right & left-leaning and straight standing).

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Probabilistic cellular automata

2. In an array flip each off-boundary 1-cycle independently w.p. $1/2$. This gives a random map F_{array} on the configurations with a given boundary. Independent sequences of F -maps define the PCAs:

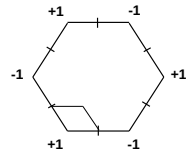
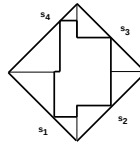
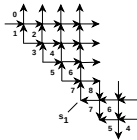
- $\mathbf{Z}^2$: $\{F_e, F_o\}$.
- $\mathbf{T}$: $\{F_{\triangle}, F_{\nabla}\}$.
- $\mathbf{K}$: $\{F_{\triangle}, F_{\nabla}, F_{hex}\}$.
- **3.4.6.4**: $\{F_{\triangle}, F_{\nabla}, F_{hex}, F_{l-\diamond}, F_{r-\diamond}, F_{s-\diamond}\}$.

These PCAs, **the Dynamic Ice models**, are irreducible and aperiodic Markov Chains hence **ergodic**. In all our runs they relaxed to the equilibrium i.e. to the measures of maximal entropy at an exponential rate.

Frozen areas

If the boundary configuration is such that the flux into the domain is zero then there is a fill-in.

Call the discrete derivative of the height along a contiguous boundary segment the **tilt** of the segment. A boundary segment of maximal tilt ± 1 uniquely determines the configuration in an interior wedge.



Max tilt for $\mathbf{Z}^2$, wedges in a diamond and a frozen corner lozenge for $\mathbf{T/K}$.

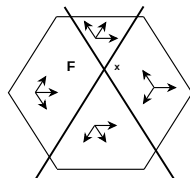
Entropy

The **entropy of a boundary condition** for a diamond/hexagon denotes the exponential size of the set of its legal fill-inns: $h_N = \frac{\log(\# \text{ of configurations})}{\# \text{ of arrows}}$, where $\#$ denotes the number on the entire domain.

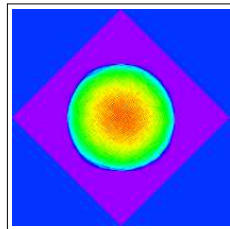
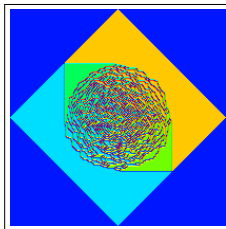
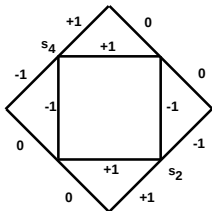
Proposition

On the square, triangular and Kagomé lattices the entropy can attain arbitrarily small positive values in the scaling limit.

For $\mathbb{K}$, $\mathbb{T}$ consider the seed configuration on the right. Subdomain F will remain frozen and the rest will have positive density of say even 1-triangles. Moving x rightwards gives arbitrary pos. upper bound for entropy. On $\mathbb{Z}^2$ the argument is analogous.

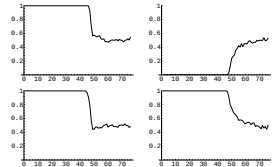
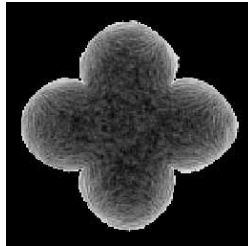
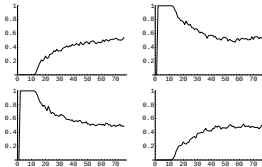


Arctic geometry in a $\mathbb{Z}^2$ diamond



Ridge roof boundary, vertex configuration and cumulative flip count.
102-diamond, equilibrium snapshot at 20.000 (rendered from 1-square
array), followed by 5.000 iterate flip collection. (DWBC)

Z^2 sectional probabilities

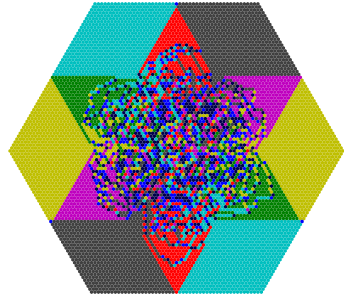
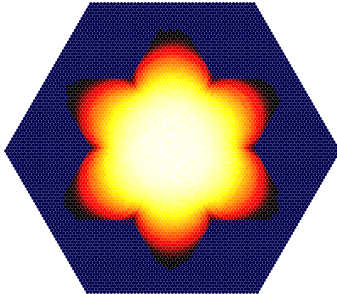


Horizontal and diagonal sections of orientation probabilities for 1-square side arrows. 75-diamond, 2500 iterates at equilibrium. Flip density at the center (diamond rotated by 45 degrees).

Arctic geometry from spatial phase transition

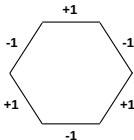
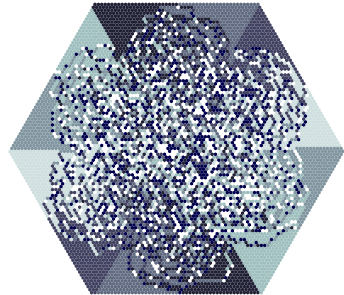
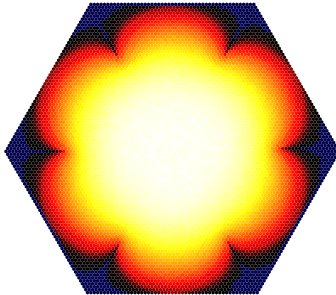
- There is strong numerical evidence for the **Arctic Circle phenomenon** to take place in Archimedean Ice models. The notion is originally from dimer context and due to Jockush, Propp and Shor.
- In the spatial scaling limit (lattice spacing $\rightarrow 0$ in a unit polygon) boundaries with extremal tilt segments yield unique phase transition curves in the interior. A variational principle proof for the $\mathbb{Z}^2$ dimers (Cohn, Kenyon and Propp, later generalized to bipartite graphs by Kenyon, Okounkov and Sheffield) can be extended to cover $\mathbb{Z}^2$ Ice.
- Neither the domain shape nor the boundary configuration needs to be finely tuned for the sharp separation of frozen and tempered subdomains of the interior to take place.
- Triangular and Kagomé lattice Ice behave in a similar way.

Arctic geometry on T/K



Cumulative flip count and vertex configuration snapshot at equilibrium on T .
(99-hexagon, $10^5 + 10^5$ iterates at equilibrium, boundary cond. as before.)

Arctic geometry on $\mathbf{T}/\mathbf{K}$



Boundary tilt, cumulative flip count and
filtered configuration at equilibrium on $\mathbf{T}$.
(99-hexagon, $10^5 + 2 \cdot 10^5$ iterates.)

Extremal tilt on 3.4.6.4.

From the periodicity properties of 3.4.6.4. lattice one can derive

Proposition

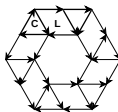
Consider a 3.4.6.4. configuration on a hexagon and a n -block of consecutive boundary arrows along any of its straight edges. If the boundary arrows are of period eight in such a block the height over the block satisfies $|\Delta h| \leq (3n + 7)/4$. For an arbitrary n -block of arrows, $n \geq 15$, the bound is $|\Delta h| \leq (13n + 28)/15$. Hence if the boundary height exists in the scaling limit and has tilt, the absolute value of the latter cannot exceed $13/15$.

Now there are **no boundary assignments of arrows with extremal tilt ± 1 .**

Entropy

Proposition

In any 3.4.6.4. Ice configuration in the set of 1-triangles and 1-lozenges at least $1/7$ of them are unidirectional. If the scaling limit entropy for a given boundary exist, it is bounded from below by $\frac{1}{24} \log 2$.



Designing any low entropy boundary is hard: $\frac{1}{6} \log 2$ can be found, seed as in the center.

No-go

For all 3.4.6.4. boundaries checked the equilibrium configurations appeared rather homogeneous. This agrees with the “No-go”-result:

Theorem

There are no frozen configurations in 3.4.6.4. Ice. The low entropy result of the three other Archimedean Ice fails for 3.4.6.4. No spatial phase transition in the scaling limit and no Arctic curves. No Archimedean universality.

References

[E1] Eloranta, K.: Diamond Ice, *J. of Stat. Phys.* **96**, 5/6, pp. 1091-1109, 1999, www.math.hut.fi/~kve/research.html.en.

[E2] Eloranta, K.: Archimedean Ice, 23 pp., 2009, submitted, [arXiv:math-ph/0909.4007](https://arxiv.org/abs/math-ph/0909.4007).

Thank you!