

Sequences with long range exclusions

Kari Eloranta

Institute of Mathematics
Aalto University, Finland

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The problem

Definition

Consider the spaces of infinite 1-dimensional sequences of symbols from $S = \{1, 2, 3, \dots, d\}$ with an **exclusion rule**:

$$(1) \quad X_{(d,f)} = \left\{ x \in S^{\mathbf{Z}} \mid x_i \neq x_{i+f(n)}, i \in \mathbf{Z}, n \in \mathbf{N} \right\}$$

where $f : \mathbf{N} \rightarrow \mathbf{N}$ is a strictly increasing function.

One-sided case $X_{(d,f)}^+$: $\mathbf{Z}$ in (1) replaced by $\mathbf{N}$.

Basic questions: When is $X_{(d,f)}$ non-empty? Can it be of exponential size? What are the generic elements like? If only finite sequences, what are they like?

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Examples, linear f

$S = \{1, 2\}$ and $f(n) = 2n$.

$x_0 = 1$ implies $x_{2k} = 2, \forall k \neq 0$. But $x_2 = 2$ implies $x_{2m} = 1, \forall m \neq 1$, a contradiction. So $X_{(2,2n)} = \emptyset$.

In fact $X_{(d,kn)} = \emptyset$ for all $d, k \geq 2$. Just exhaust S :

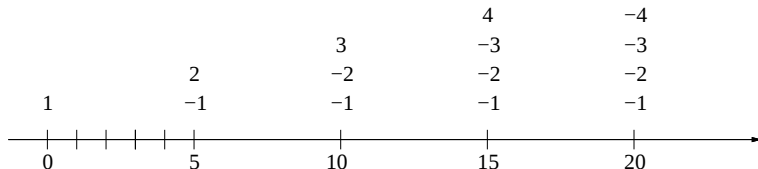


Figure: $X_{(d,5n)} = \emptyset$.

But for $S = \{1, 2\}$ and $f(n) = 2n - 1$ we have periodic points $(12)^*$, hence $X_{(2,2n-1)} \neq \emptyset$. Favourable parity!

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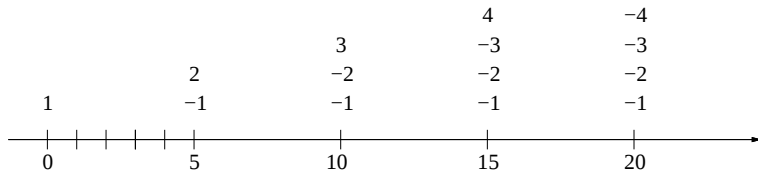


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Examples, faster growing f

$S = \{1, 2\}$ and $f(n) = n^r$, $r = 2, 3, \dots$

If $x_0 = 1$ then $x_{2i} = 1$, $\forall i \in \mathbf{Z}$ so in particular $x_{2^r} = 1$, a contradiction.

Therefore $X_{(2, n^r)} = \emptyset$.

Suppose there is $m \in \mathbf{N}$ which does not divide any of the values $f(n)$, $n \in \mathbf{N}$.

Then for $d \geq m$ we can have periodic points.

For example $X_{(3, 2^n)}$ and $X_{(4, \{\text{primes}\})}^+$ are nonempty.

$X_{(d, n!)}^+$, $d = 2$ or 3 can immediately be seen to be empty. But ...

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Non-trivial example

... $X_{(4,n!)}^+$ could be non-trivial. There is a period (of length 25) which repeats almost until the exclusion would violate it for the first time at 5041.

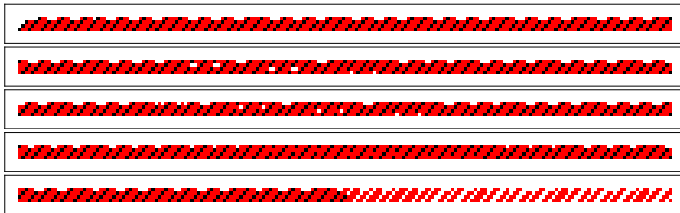


Figure: Lexicographically generated candidate for $X_{(4,n!)}^+$ (from $x_1 = 1$). Segments, from top: 1-200, 4950-5150, 10000-10200, 362950-363150, 499900-500100.

Periodicity contradicted at intervals of length $n!$, $n = 7, 10, 11, 12 \dots$
 but the sequence generation survives them at least half a million steps.

Languages

Theorem

If for any natural m there is a natural n such that we have $m \mid f(n)$ then the words satisfying the exclusion do not form a context-free language. Hence the sequences do not form a regular language (sofic shift) either.

Proof by showing that the validity of the appropriate Pumping Lemma is dependent on the (non)divisibility property.

Powers

Consider $d \geq 3$ and $f(n) = n^r$, $r = 2, 3, \dots$

- None of the corresponding languages are context-free (by the Theorem)
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- $X_{(3,n^2)}^+$ and $X_{(3,n^2)}$ are empty (elementary argument).
- $X_{(4,n^2)}^+ = \emptyset$ by a computer assisted proof. Max sequence length is 47.
- For $d = 5$ one can generate sequences of length at least 170.
- Random generation of sequences for $X_{(d,n^2)}^+$, $d = 5, 6, 7, 10, 15$ and 20 suggest strongly that all these spaces are empty.
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Additive combinatorics

For $A \subset \mathbf{N}$ let $A - A = \{a - a' \mid \text{any } a, a' \in A\}$ and $A^{(N)} = A \cap \{1, 2, \dots, N\}$.

Question: If we insist that $f(n) \notin A - A$ for any natural n , what is A like?

For n^2 (Lovász's conjecture) Furstenberg and Sárközy showed in 1977-8:

Theorem

Given $\delta > 0$ there is $N_0(\delta)$ such that if $N \geq N_0(\delta)$ and $|A^{(N)}| \geq \delta N$ then there is natural n such that $n^2 \in A - A$.

The proofs were ergodic theoretic and Fourier analytic respectively.

In 1994 Balog, Pelikán, Pintz and Szemerédi proved

Theorem

For any natural $k \geq 2$ if $n^k \notin A - A$ for all n then $\frac{|A^{(N)}|}{N} \ll \frac{1}{(\log N)^c \log \log \log \log N}$.

($\ll$ means “less than constant times”. Best c has been worked out: $1/\log 3$.)

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Additive combinatorics applied

Given d symbols and $f(n) = n^k$, set $A_i = \{ \{j\} \mid x_j = i \}$ and let A_i , $i = 1, \dots, d$ partition $\mathbf{N}$ (1-sided sequence defined everywhere). Then the last Theorem in particular implies that if the exclusion is to hold for the sequence for the given f , for sufficiently large n the densities of A_i 's cannot add up to 1, a contradiction. Hence all $X_{(d,n^k)}^+$ and therefore all $X_{(d,n^k)}$ are empty, $k = 2, 3, \dots$

The BPPS result has been extended for **intersective** polynomials: $f \in \mathbf{Z}[x]$ s.t. $f(n) \equiv 0 \pmod{q}$, $\forall q$ natural, exactly the same condition as in the language characterization.

...so either the sequences try to be complicated and surely die or are simple i.e. periodic and can live forever!

Note: $n!$ grows superexponentially and is not included in the results above.

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Algorithm for one-sided sequences

Algorithm v2.0:

0. set $M \geq 1$, let $S_j = S$ at each $j \in \{1, \dots, M\}$ and set $i = 1$.
1. if $S_i = \emptyset$ then **halt**,
 else pick uniformly a random symbol $s \in S_i$.
2. update $S_j \leftarrow S_j \setminus \{s\}$ for all $j = i + f(n) \in \{i + 1, \dots, M\}$, $n \in \mathbf{N}$.
3. if $i = M$ **halt** and call **full length**,

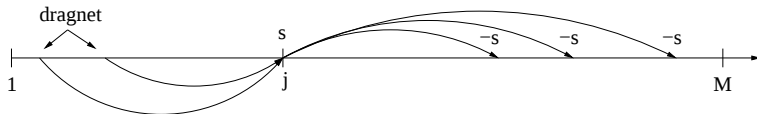
i.e. each coordinate is chosen independently and uniformly but in such a way as to respect the restrictions from all the relevant coordinates in its past.

Probabilistic model

Dragnet D_j is the set of coordinates less than j restricting the assignment at j . Its cardinality is a step-function, equal to d from the start of the first **interval**.

For $f(n) = n^2$ this is at coordinate $j = d^2 + 1$ and the i^{th} interval is from $(d + i - 1)^2 + 1$ to $(d + i)^2$ (length $l_i = 2(d + i) - 1$).

If the sites on the dragnet D_j support the entire alphabet S then at site j there is a **full block**. First full block is possible at the start of the first interval.



Probabilistic model

Assume that all the symbols on $\{1, 2, \dots, j-1\}$ have been laid out independently and uniformly from S . Then

Proposition

Let B_j be the event that one has the first full block at j in the i^{th} interval. Then

$$(2) \quad \mathbf{P}(B_j) = p_i = \frac{1}{d^{d+i-1}} \sum_{\substack{k_r \geq 1, \ r=1, \dots, d \\ k_1 + \dots + k_d = d+i-1}} \binom{d+i-1}{k_1 \ k_2 \ \dots \ k_d}$$

where the sum is d -fold over the given positive integers.

Recall the multinomial: $\binom{a}{b_1 \ b_2 \ \dots \ b_d} = \frac{a!}{b_1! b_2! \dots b_d!}$, $\sum_{i=1}^d b_i = a$.

Proof is just combinatorics on the dragnet.

Probabilistic model

On the interval with dragnet cardinality $d + i - 1$ the sequence extension halts w.p. p_i and its length on the interval $\sim \text{Geom}(p_i)$.

Lemma

For an alphabet S of size d one has for all $i \geq 1$

$$1 - p_i < d \left(1 - \frac{1}{d}\right)^{d-1} \left(1 - \frac{1}{d}\right)^i.$$

For the proof of the Lemma one has to consider the entries on the $(d + i - 1)^{\text{th}}$ level (from the top) of Pascal's d -simplex. Multinomial Theorem gives the total sum but for $1 - p_i$ we need to bound its boundary sum. Note that $p_i \uparrow 1$ is obvious, but its geometric lower bound requires some work.

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Theorem

Let the assumptions on the sequence be as above and $f(n) = n^2$. Then a full block materializes at j i.e. $\mathbf{P}(\text{sequence generation halts at } j) =$

$$\begin{cases} 0 & 1 \leq j \leq d^2 \\ (1 - p_1)^{[j-d^2-1]} p_1 & j \text{ in the first interval} \\ \left(\prod_{k=1}^{i-1} (1 - p_k)^{l_k} \right) (1 - p_i)^{[j-d^2-1 - \sum_{k=1}^{i-1} l_k]} p_i & j \text{ in the } i^{\text{th}} \text{ interval, } i \geq 2 \end{cases}$$

and the halting time distribution has a geometric tail.
 The sequences generated are almost surely of finite length.

The Theorem follows by combining the geometric halting probabilities on the intervals, "uniformizing" them for a tail estimate (l_k are not equal) and finally using Borel-Cantelli.

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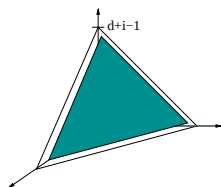
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Remarks

- The sum

$$\sum_{\substack{k_r \geq 1, r=1, \dots, d \\ k_1 + \dots + k_d = d+i-1}} \binom{d+i-1}{k_1 \ k_2 \ \dots \ k_d}$$

has asymptotically an exponential number of summands both in d and i . To use the Theorem for large d and i one needs to find an efficient way to compute the p_i 's.



For small i the sum can be compressed. E.g. for $i = 4$:

$$\binom{d}{1} \binom{d+3}{1 \ \dots \ 1 \ 4} + 2 \binom{d}{2} \binom{d+3}{1 \ \dots \ 1 \ 2 \ 3} + \binom{d}{3} \binom{d+3}{1 \ \dots \ 1 \ 2 \ 2 \ 2}$$

but this gets complicated soon... Estimates for the tail if $i \gg 1$.

Remarks

- While $p_i \uparrow 1$ monotonically, the halting distribution is jagged:
 At the i^{th} jump

$$\frac{\mathbf{P}(\text{halts at } (d+i)^2 + 1)}{\mathbf{P}(\text{halts at } (d+i)^2)} = \frac{1-p_i}{p_i} p_{i+1} \rightarrow 0 \quad \text{as } i \rightarrow \infty$$

but far exceeds 1 earlier.

- The model applies verbatim at least to all n^k , $k = 2, 3, \dots$
- The independence assumption seems heavy for small alphabet but less so for a large one. But actually...

Reality check for n^2 and $d = 5, 10$ and 15

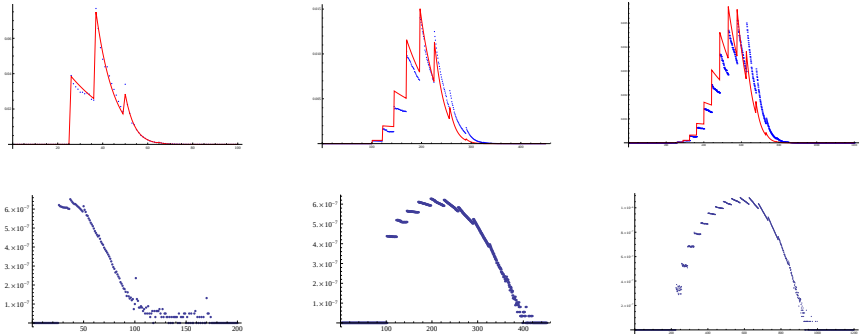


Figure: Top row: empirical (blue/rough) and theoretical (red/smooth) halting probability distributions. Bottom row: log of that above for the data.

Dependencies, squared

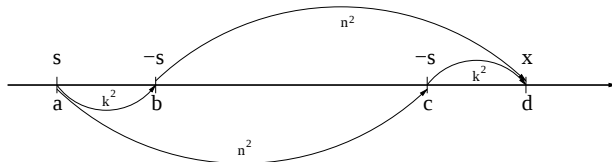


Figure: A dependency mechanism affecting the termination probability. Dragnet at d .

$$\mathbf{P}\left(\text{seq. term. at } d \mid k^2 + n^2 \neq m^2\right) < \mathbf{P}\left(\text{model term. at } d \mid k^2 + n^2 \neq m^2\right)$$

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As k and n vary, the non-square case is far more likely to occur than the square case. So termination probabilities of the independent model should major the observed ones.

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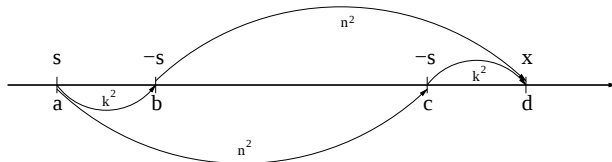


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Statistics of the sequence lengths for n^2

Symbols d	Empirical mean	Empirical std. dev.	Model mean	Model std. dev.	Sequences
4	27.2542	5.13374	23.992	5.23924	$50 \cdot 10^6$
5	39.5672	8.28983	39.2172	8.22516	$80 \cdot 10^6$
6	60.8247	13.5813	59.3666	11.9713	$80 \cdot 10^6$
7	89.4687	18.5912	84.982	16.5113	$30 \cdot 10^6$
10	209.315	38.2887	199.562	35.1369	$20 \cdot 10^6$
15	566.87	92.2796	543.291	84.4349	$10 \cdot 10^6$
20	1156.57	170.829	*	*	$5 \cdot 10^6$

Table: Data from randomly generated one-sided sequences and the probabilistic model. Asterisks are due to missing coefficients (for i large).

Conjecture

Based on the above one might venture to...

Conjecture

Consider the case of d symbols and $f(n) = n^2$. Suppose $T^{(d)}$ is the halting instant of the Algorithm v2.0. For sufficiently rapidly growing $M(d)$ there are positive constants a and b such that as $d \rightarrow \infty$

$$\mathbf{P} \left(\frac{T^{(d)} - ad^{5/2}}{bd^{15/7}} \leq x \right) \rightarrow \Phi(x) \quad \forall x \in \mathbf{R}$$

where Φ is the cumulative distribution function of the standard normal $N(0, 1)$.

Such CLT should hold for the probabilistic model as well (with parameters but not exponents adjusted)

$M(d)$ just needs to outgrow the off-set rate $d^{5/2}$.

Termination details for n^2

One can record when the upcoming termination can be seen for the first time (x-coord.) and how far ahead it will be (y):

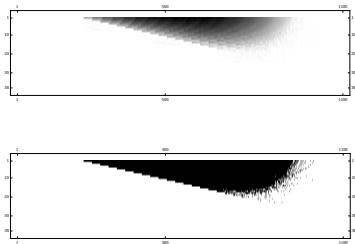
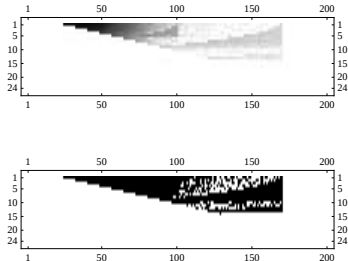


Figure: Terminal jump distribution (log and sign, top and bottom resp.) for $d = 5$ and 15 (20 and 10 million samples).

Termination details for n^2

- Left slope is due to the one-sidedness: exactly quantifiable.
- For low values ($d \leq 7$ or so) some kind of number theoretic constraints rule the behavior of the Algorithm hence the interior and the right edge (it is not a boundary effect).
- **Beyond this range the termination seems essentially like a random process.**
- The expected termination instant is far, far below the threshold coming from the BPPS theorem!

Rest

arXiv:math-ph/1204.3439
or
www.math.hut.fi/~kve/research.html.en

Thank you!



A **context-free language** is recognized by a non-deterministic pushdown automaton. Such language necessarily satisfies a **Pumping Lemma**:

Lemma

Any sufficiently long string s , say $|s| \geq k$, can be written as $s = uvxyz$ such that

- (i) $|vxy| \leq k$,*
- (ii) $|vy| \geq 1$,*
- (iii) uv^nxy^nz is an allowed string for all natural n .*

If either v or y vanishes but the other is non-trivial (hence (ii) is still valid) this reduces to the Pumping Lemma of **regular languages** (languages recognized by a finite state automaton).

If $p \nmid a$ and the congruence $x^2 \equiv a \pmod{p}$ is soluble then a is called the **quadratic residue modulo p** .

Mod $p = 2$ every integer is a quadratic residue. Let $P = \{1, 2, \dots, p-1\}$. The basic distribution result is

Lemma

Let p be an odd prime. Then exactly half of the integers a on P are quadratic residues modulo p .

Little is known on the distribution of the residues beyond this. 1 is quadratic residue and so is a if a is a square. Maximum number of residues between non-residues is $2\sqrt{p} + 1$. If $N(p)$ is the smallest non-residue in P then for large p , $N(p) < p^{1/4+\epsilon}$ (by Burgess, -57). If RH holds then $N(p) = c(\ln p)^2$.

Initializations:

```
(* n2 blocks *)
d = 4; (* number of symbols *)
M = 200; (* length of sequences attempted, must be bigger than imax! *)
blocks = Table[ If[IntegerQ[Sqrt[i]] == True, 1, 0], i, M]; (* block sites *)

(* ab-array initialization *)
ab = Table[0, d, M];
ab[[1]] = Flatten[Prepend[Drop[-blocks, -1], 1]];
ab[[2]] = Flatten[Prepend[Drop[-blocks, -2], 0, 1]];
col[i] := ab[[1, i]], ab[[2, i]], ab[[3, i]], ab[[4, i]] (* i th column of ab *)

(* for FULL RUN for |S|=4, minimal output! *)
i = 3; imax = 100; (* max seq. length constructed *)
base = 0; (* running assumption for the third column *)
maxlength = 0; (* initialization for the maximal length sequence found *)
lowbacktrack = 10; (* highest index from which backtrack is notified *)
```

... and the code...

```

(Label[fwd];
While[ i <= imax,
maxlength = Max[maxlength, i];
locs = Intersection[ Flatten[Position[col[i], 0]],
base + 1, base + 2, base + 3, base + 4]; (* free symbols above base *)
If[Length[locs] == 0, loc = 0, loc = Min[locs] ]; Label[jumpup];
If[loc == 0,
i = i - 1; If[i <= lowbacktrack, Print["cannot assign, will backtrack to: ", i] ];
Goto[backtr];,
ab[[loc, i ]] = 1; base = loc;(* new symbol assignment *)
blockcols = Flatten[Prepend[Drop[blocks, -i], Table[0, i]]];
ab[[loc]] = ab[[loc]] - blockcols; (* assigning the new blocks *)
(* check if full blocks formed *)
Do[ If[ Flatten[ Position[col[Flatten[ Position[blockcols, 1] ]][chkcol]], 0]] == ,
ab[[loc, i]] = 0;
ab[[loc]] = ab[[loc]] + blockcols; (* taking the new assignment and blocks away *)
If[ loc < Max[locs],
loc = locs[[Flatten[Position[locs, loc]][[1]] + 1]]; Goto[jumpup];,(* try higher symbol *)
i = i - 1; Goto[backtr]; ],
chkcol, Length[Flatten[ Position[blockcols, 1]] ]; i = i + 1; base = 0; ]; Abort[];
Label[backtr];
If[i == 3 && ab[[3, 3]] == 1,
Print["All done, furthest assignment: ", maxlength]; Abort[];]
If[i <= lowbacktrack, Print["recalling assignment at: ", i] ];
base = Flatten[Position[col[i], 1]][[1]]; (* position of the assignment to be recalled *)
blockcols = Flatten[Prepend[Drop[blocks, -i], Table[0, i]]];
ab[[base, i]] = 0;
ab[[base]] = ab[[base]] + blockcols; (* taking the assignment and its blocks away *)
If[base == d, i = i - 1; Goto[backtr];, Goto[fwd]; ]

```