Entropy Geometry in Vertex Models

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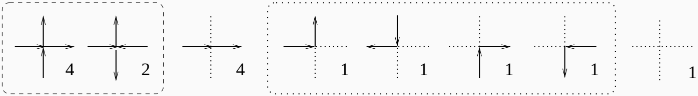
Vertex Models

Vertex Models

Definition (Zero Flux Vertex Model on the Square Lattice)

Assign arrows or unoriented edges between a vertex and its nearest neighbors in $\mathbf{Z}^2$ *respecting the* **rule***: there is the same number of incoming and outgoing arrows. The model consists of all configurations in which the rule is satisfied at every lattice point.*

No spin variables in these models. Three well known examples:



Dotted line segment means unoriented edge. Numbers are multiplicities.

Six-vertex (Ice) rule: The frame on the left.

15-vertex rule: The entire set.

19-vertex rule: Remove the asymmetry of the 15-vertex rule: add the four missing orientations of the central frame.

Past and aim

Six-vertex rule has been intensely studied and serves here as a reference model. For preceding work on the 15/19-vertex models, see e.g. Izergin & Korepin, Fateev & Zamolodchikov (-81), Batchelor (with twisted boundary condition, -91), Inami & Odake & Zhang (-96), Pant and Wu (knot invariant, -97), recently Garbali, Hagendorf and others.

We will study the models on a simple bounded domain with a special boundary condition that will make comparison to Ice Model/Alternating Sign Matrix and still earlier dimer results transparent.

In Ice one has a fully packed loop soup and all entropy arises from a single action, the reversal of unidirectional loops. 15/19-vertex ones relax the packing and allows a **diluted loop soup**. What does that imply in terms of limit shapes? Because of multiple random actions each with its own distribution within the domain a more involved **entropy geometry** appears.

Static bits

Definition

Height function *is a mapping from dual lattice* $\mathbf{Z}^2 + (\frac{1}{2}, \frac{1}{2})$ *to integers. If one crosses a lattice arrow pointing left, it increases by one, if to the right, decreases by one and if no arrow encountered, height stays constant. Given a vertex configuration on a simply connected domain it is unique up to an additive constant. The graph of the height function is a Lipschitz-surface over such configuration (***tilt** *extrema* ± 1*).*

Definition

If we have two configurations c *and* c' *on the same domain having the same boundary configuration,* C *and* C' *are the solids under their height surfaces and* $c \succeq c'$ *if* $C \supseteq C'$ *then* $\succeq$ *defines a* **partial order** *on the configurations.*

Proposition

Such configurations form a **distributive lattice***.*

($(L, \vee, \wedge)$ is s.t. $\forall x, y \in L: \; x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$ and $(\wedge \rightleftharpoons \vee)$.)

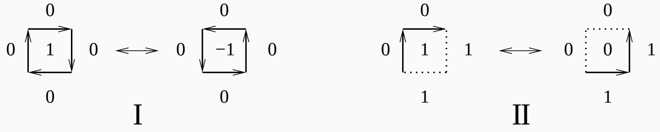
Dynamical Models

Dynamical Models

For the vertex models here any unidirectional loop or unidirectional infinite path can be reversed to generate a new configuration. In 19-vertex model additionally any such loop/path can be converted undirected or vice versa.

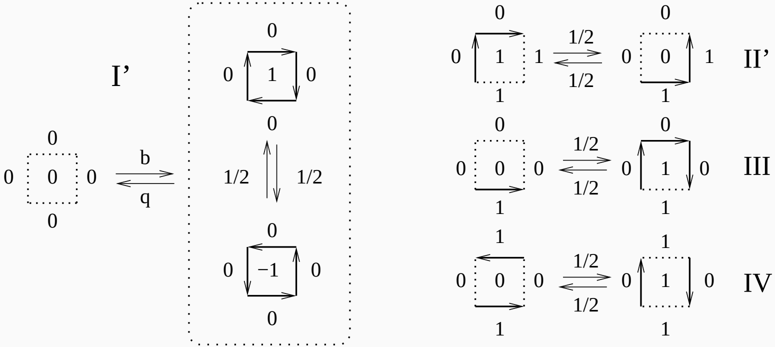
To implement an efficient algorithm one should do the allowed perturbations using the smallest legal **actions/local moves/flips**.

The left action I suffices to generate all six-vertex configurations.



I and II together with rotation by π generates for 15-vertex rule.

The minimal set of local moves for the 19-vertex rule (up to rotation and reflection):



Heights noted around and inside squares, transition probabilities next to arrows. All reflections and rotations must be included.

Clearly the three action sets cannot generate illegal configurations from legal ones i.e. these are necessary in each case. Conversely by careful study of the height surfaces one can show that these suffice.

One can utilize a **deposition/sublimation** model. Depending on the case one uses one or two types of volumes: unit cubes or upright $1 \times 1 \times 2$ -pieces. With these one can fill local minima, cut down local maxima and reach the maximal/minimal element for the given boundary condition. Irreducibility of the configuration sets under the action sets follow.

Theorem (Irreducibility and ergodicity)

Given a 19-vertex configuration on a bounded domain, any other legal configuration with the same boundary condition can be generated from the former using a finite sequence of elementary actions I'-IV. A strict subset of actions will not suffice. When $0 < b, q < 1$ the Markov Chain on the graph of legal configurations is ergodic.

The conclusions hold for the subsets of actions of six-vertex and 15-vertex rules. A unique stationary measure exists for all.

Computation bits

The actual computation of 6/15/19-vertex model is most naturally done on a $N \times N$ array of symbols each coding the arrows around a lattice unit square (so $2^4 = 16$ or $3^4 = 81$ symbols in the alphabet). This array is a diamond enclosing the domain square.

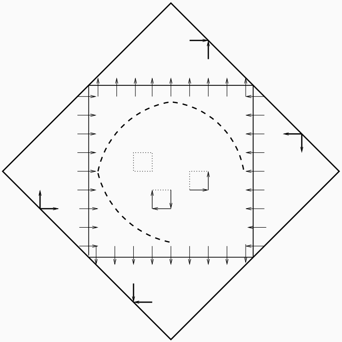
The boundary condition is imposed on the diamond, specifically on its $4 \times 2N$ lattice edges. When done right it will force a desired boundary condition in the inscribed $N \times N$ square in all the models.

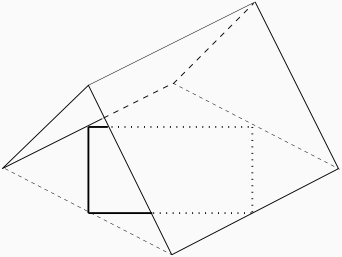
The Probabilistic Cellular Automaton (PCA) which implements the MC acts on the diamond array split as a checkerboard. The black and white subsets are updated alternatively. On each color the updating is done according to the MC, sitewise independently, followed by the update of the other color subset for consistency.

Boundaries

DWBC

For the sake of simplicity and comparisons we restrict to square domain. The Izergin-Korepin **Domain Wall Boundary Condition** or its relaxation are used:





DWBC is on the inscribed square. The diamond is due to computation alone. On the right the **ridge roof** height. It forces DWBC inside.

Proposition

In a 15-vertex configuration an unoriented interior edge implies unoriented boundary edges. Hence with DWBC the 15-vertex rule reduces to the six-vertex rule.

Next we'll show how to relax DWBC to get genuinely (non-six-vertex) 15-vertex configurations.

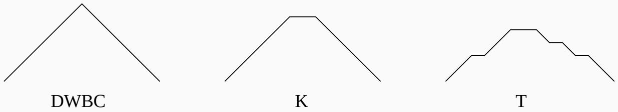
Proposition

For a given boundary condition the number of unoriented edges in the 15-vertex fill-in configuration is constant.

This is very far from true for 19-vertex configurations. In that context DWBC can bound highly non-trivial, very non-six-vertex configurations.

Non-DWBC for 15-vertex

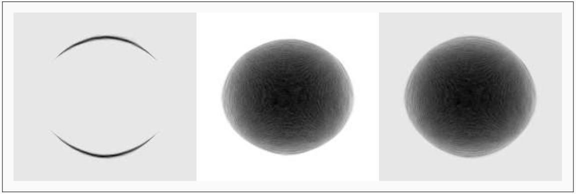
Two generalizations of the Ridge roof (left, imposes DWBC):



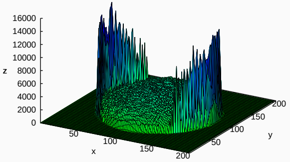
Cross cuts above are NW-SE sections of the initial height over the diamond. The flat bits correspond initially to unoriented SW-NE staircases. Their end points are fixed on the boundary and under the iteration behave like non-branching random curves stretching across the diamond.

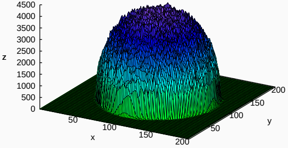
Samples

15-vertex equilibria, non-DWBC

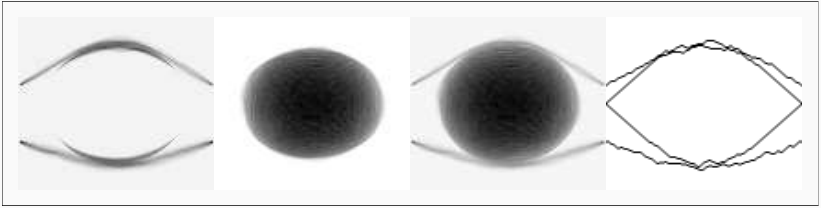


K-boundary, a pair of blank staircases at the top. Diamond tilted by $\pi/4$; ridge is horizontal. Densities of II, I and I+II. 206^2 diamond/square, equilibrium iterates 61-70.000. Below left II weighted 10-fold.





Equilibrium from T-type boundary condition for 15-vertex model:

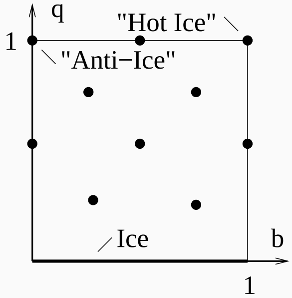


Cumulative distributions of II, I and I+II. On the right the locations of the unoriented edges at termination. Square 106^2, ridge horizontal, 10.000 iterates at the equilibrium.

19-vertex parametrization (and back to DWBC)

- Bottom (all b) is six-vertex since no arrow vacancy can come about. 1-enumeration point of ASM in the disordered phase. Limit shape a bit off circle.
- “Anti-Ice”: no trace of Ice-action since no unidirectional loops are reversed.
- $b + q$ is the unidirectional 1-loop creation-annihilation rate, “Ice-temperature”.

• Sample points.



DWBC and density

DWBC (which is just "maximal tilt boundary segments without cul-de-sacs") can be generalized to more complicated domain shapes yielding more complicated limit shape results to 6/19-vertex.

On the boundary the arrow density is one, but

Proposition (Interior arrow density)

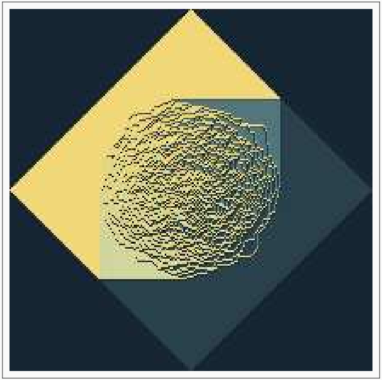
The arrow density of a 19-vertex configuration over a domain with DWBC is always at least $1/2$*. Bound is tight.*

We do not know of any boundary condition with density of boundary arrows strictly below one yielding non-trivial limit shapes.

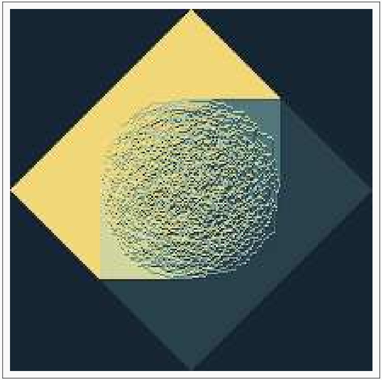
Ergo, all subsequent 19-vertex samples are with DWBC at equilibrium.

From Ice to Hot-Ice

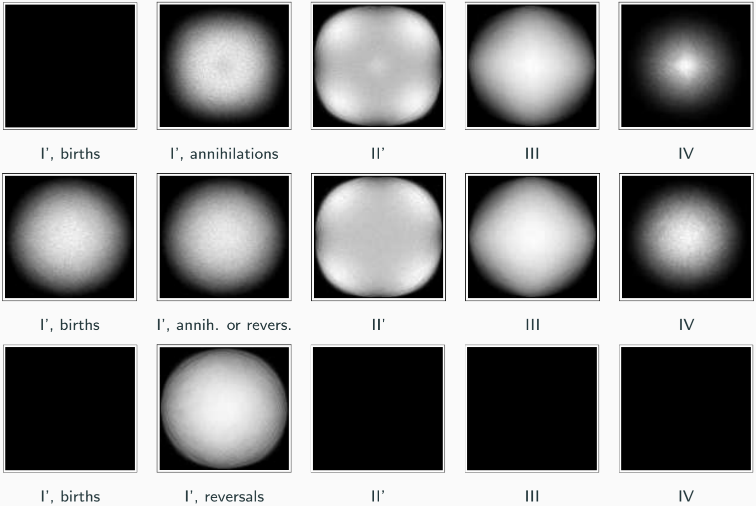
Configurations after 40.000 iterates with DWBC on the maximal square inside 106^2 diamond/square. 1-square arrow arrangements color coded.



Ice $((b,q) = (0,0))$

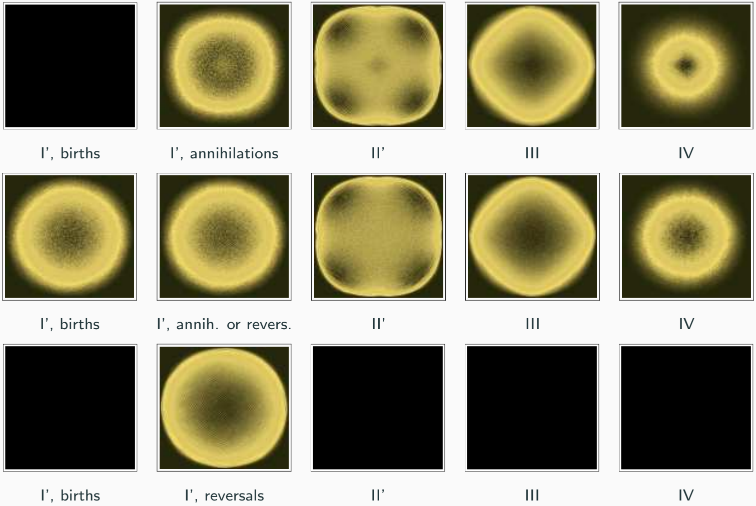


Hot-Ice $((1,1))$



Action densities at equilibrium. Captions indicate the (sub)actions.
Top row: Anti-Ice ($(b, q) = (0, 1)$). **Middle row**: diagonal $q = b > 0$.
Bottom row: 1-weight ASM i.e. Ice-case ($(b, 0)$, any b).
106^2 square. Lighter is more active. Individually scaled for best contrast.

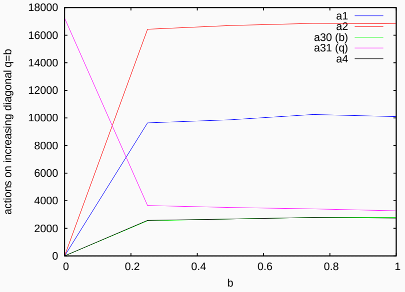
Same data a bit enhanced:



(106^2 square, samples from equilibria.)

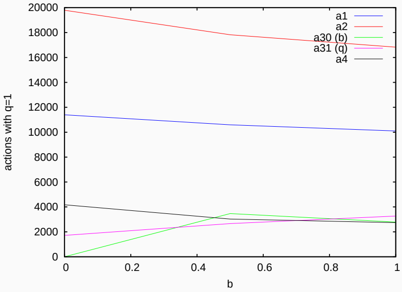
Intensities of actions on $q = b$

- Distribution maxima of actions I'-IV along the diagonal $q = b$. The graphs should be constant for $b > 0$, since on the diagonal all weights are 1.)
- The ordering of the intensities on the right, from the top: III>II'>I'(death & rev.) ≥I'(birth)≈IV.



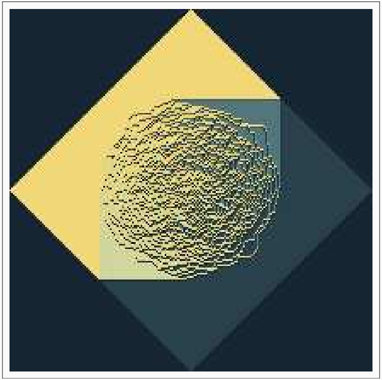
Intensities at the top

- Distribution maxima of actions I'-IV along the top, $q = 1$.
- III>II'>I'(death) ≈I'(birth)≈IV

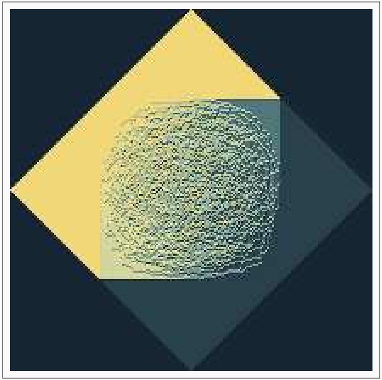


From Ice to Anti-Ice

Configurations after 40.000 iterates on 106^2 diamond, DWBC:



Ice

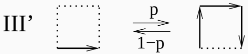


Anti-Ice $((0, 1))$

Action III' skewed

Actions I'(reversal), II' and IV do not change the arrow density.

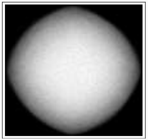
Action III changes the arrow density and it alters the geometry of locally oriented paths. Moreover it is the highest intensity action.



As $p \downarrow 0$, the oriented paths should straighten out and form thinner ensembles, hence further lowering the intensities of other actions.

As $p \uparrow 1$, the oriented paths get more convoluted, perhaps approaching lattice filling. This should even out the action distribution differences.

The effect should be most pronounced in the directions of lattice axes, but what are the other **global consequences?**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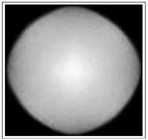
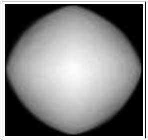
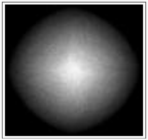
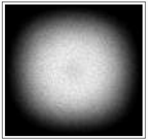
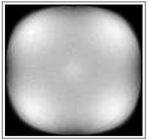
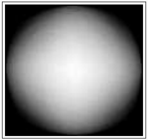
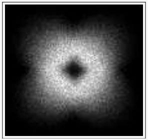

II'

III

IV

I', annih. or revers.

II'

III

IV

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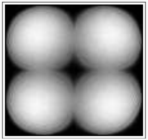
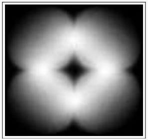
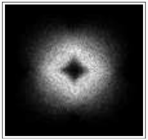
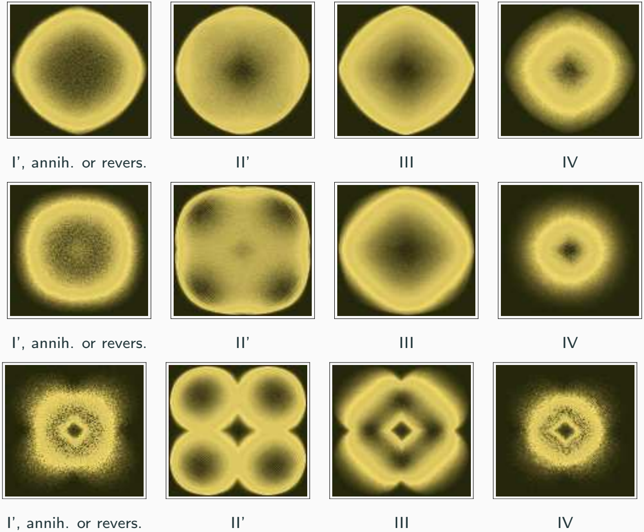
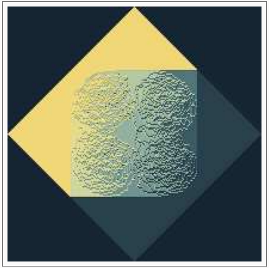

II'

III

IV

Skewed action III' at $(b, q) = (0, 1)$. Middle row is Anti-Ice ($p = \frac{1}{2}$). Top row: $p = \frac{7}{8}$ and bottom row $p = \frac{1}{8}$. No I'-births in any: first blank column removed. 106^2 square.



Skewed action III' at $(b, q) = (0, 1)$, enhanced rendering. Middle row is Anti-Ice ($p = \frac{1}{2}$). Top row: $p = \frac{7}{8}$ and bottom row $p = \frac{1}{8}$.

Thank you!



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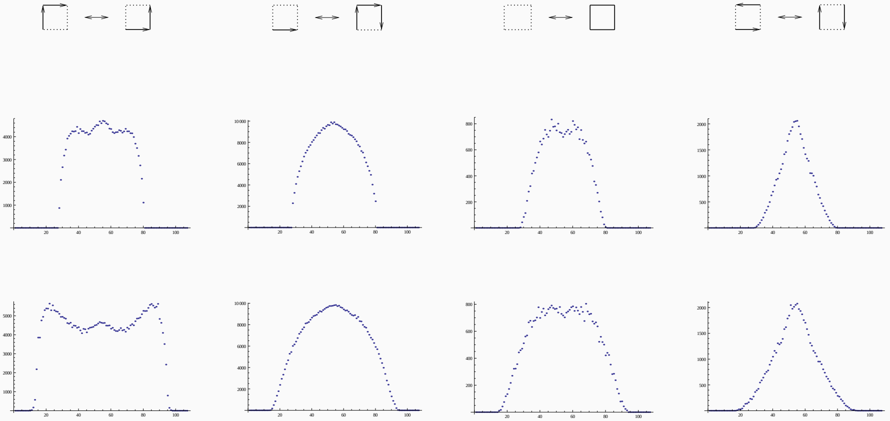
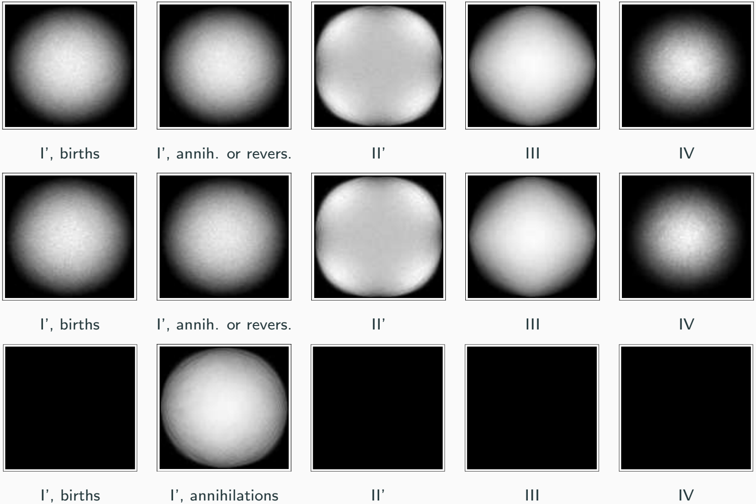
Appendices

Anti-Ice action sections

Horizontal and diagonal sections for the Anti-Ice action distributions:

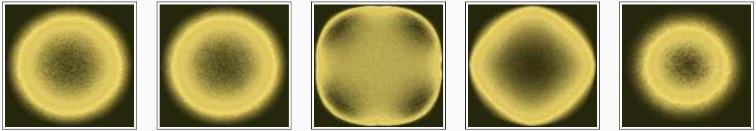


Diagonal $q = b$ action distributions

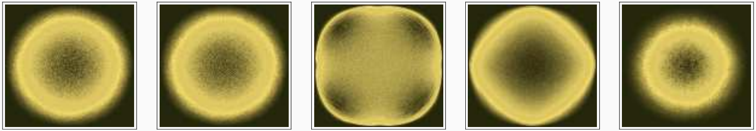


Top: $(b, q) = (1, 1)$, middle: $(\frac{1}{2}, \frac{1}{2})$ and bottom: $(0, 0)$.

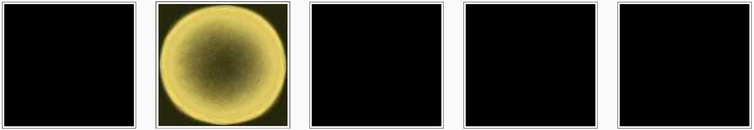
Diagonal $q = b$ action distributions enhanced



I', births | I', annih. or revers. | II' | III | IV



I', births | I', annih. or revers. | II' | III | IV

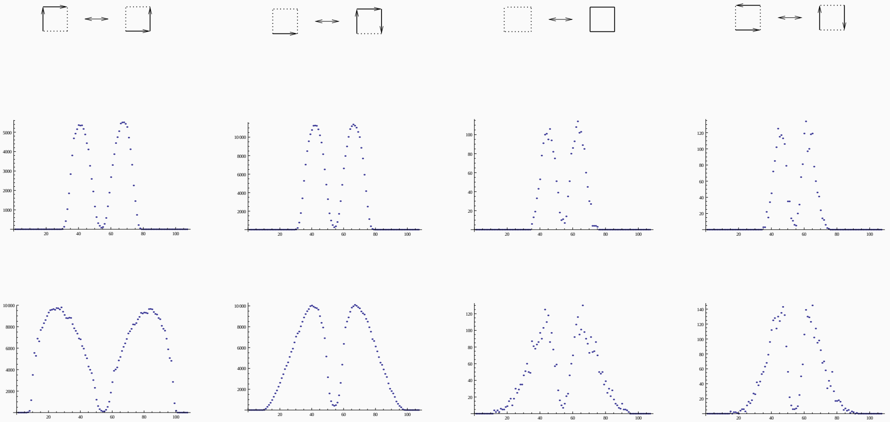


I', births | I', annihilations | II' | III | IV

Top: $(b, q) = (1, 1)$, middle: $(\frac{1}{2}, \frac{1}{2})$ and bottom: $(0, 0)$.

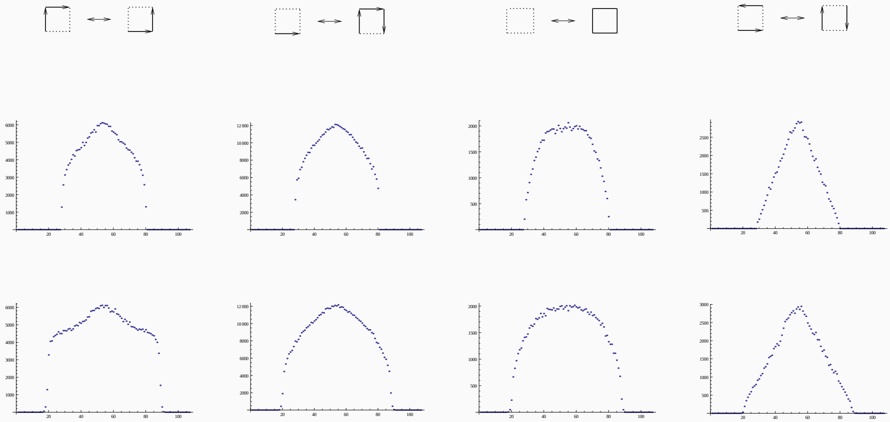
Anti-Ice distribution sections, weight $p = \frac{1}{8}$

Horizontal and diagonal sections for the $p = \frac{1}{8}$ action distributions:

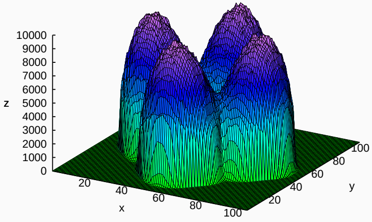


Anti-Ice distribution sections, $p = \frac{7}{8}$

Horizontal and diagonal sections for skewed Anti-Ice, $p = \frac{7}{8}$:

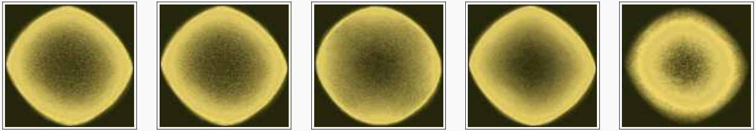


Skewed Anti-Ice actions, $p = \frac{1}{8}$

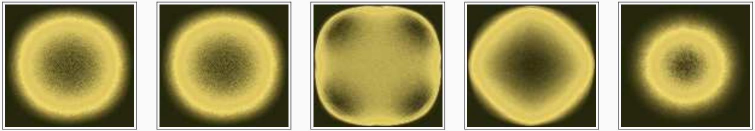


Action II' & III' distributions for Anti-Ice with $p = \frac{1}{8}$. 106^2 diamond.

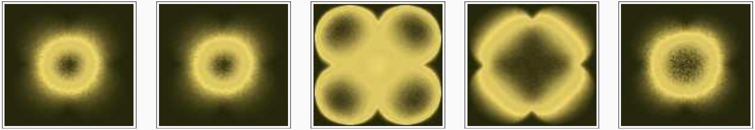
Top center $\left(\frac{1}{2}, 1\right)$ skewed



I', births | I', annih. or revers. | II' | III | IV



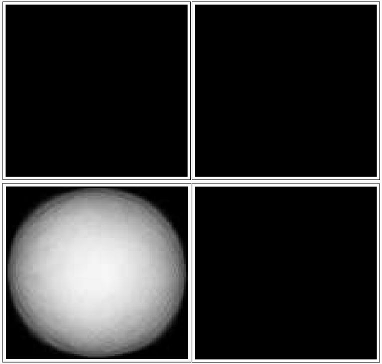
I', annih. or revers. | I', births | II' | III | IV

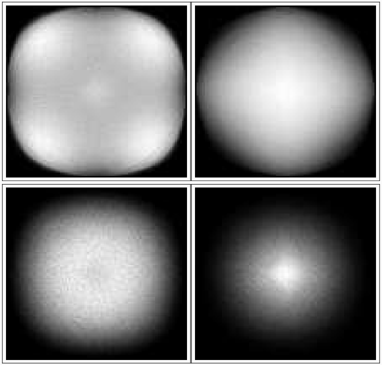


I', births | I', annih. or revers. | II' | III | IV

Middle row: unskewed action III around $\left((b, q) = \left(\frac{1}{2}, 1\right),\ p = \frac{1}{2}\right)$.
Top row: $p = \frac{7}{8}$, **bottom row**: $p = \frac{1}{8}$.

Ice versus Anti-Ice



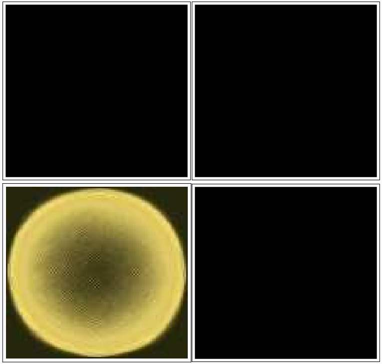


Ice action distribution

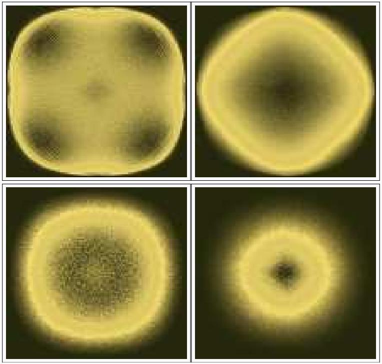
Anti-Ice action distributions

(actions on a 106^2 square at equilibrium.)

Ice versus Anti-Ice, enhanced



Ice flip distribution



Anti-Ice flip distributions

(actions on a 106^2 square at equilibrium.)